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SUPPLEMENT TO "HYPOTHESIS TESTING ON LINEAR STRUCTURES OF HIGH DIMENSIONAL COVARIANCE MATRIX"

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Section S.1 presents some results on random matrix theory. Section S.2 consists of the proof of Lemma 2.1. Section S.3 presents the proofs of Theorem 2.4 (a) and (b). Section S.4 provides the proof of Theorem 2.5. Simulation results on estimation of κ are given in Section S.5. Theorem 2.3 is from Lemma 2.1 by the delta method. Proof of (c) of Theorem 2.4 is given in the main paper.

S.1. Some results on random matrix theory. We first define the Marčenko-Pastur law. For $0 < \theta < \infty$, let $a(\theta) = (1-\sqrt{\theta})^2$ and $b(\theta) = (1+\sqrt{\theta})^2$. The Marčenko-Pastur distribution of index θ, denoted by F^θ, has the density function

$$f^\theta(x) = \begin{cases} \frac{1}{2\pi\theta x}\sqrt{[b(\theta)-x][x-a(\theta)]}, & a(\theta) \le x \le b(\theta) \\ 0, & \text{otherwise}, \end{cases} \tag{S.1}$$

and has a point mass $1-1/\theta$ at the origin if $\theta > 1$. Let $y_n = p/n \to y \in (0,\infty)$ and F^y, F^{y_n} be the Marčenko-Pastur law of index y and y_n, respectively. Let $\mathcal{U}$ be an open set of the complex plane, including $[a(y), b(y)]$ and $\mathcal{A}$ be the set of analytic functions $f : \mathcal{U} \mapsto \mathbb{C}$. We consider $G_n := \{G_n(f)\}$ indexed by $\mathcal{A}$,

$$G_n(f) = \sum_{i=1}^{p} f(\lambda_i) - p\int_0^{+\infty} f(x)dF^{y_{n-1}}(x), \quad f \in \mathcal{A}, \tag{S.2}$$

where λ_j's are eigenvalues of $\mathbf{F}$ defined in Section 2.3. Lemmas S.1, S.2 and S.3 below will be repeatedly used in the proofs of the theorems.

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LEMMA S.1. *Assume that $f_1, \cdots, f_k \in \mathcal{A}$, and $\{w_{ij}\}$ is a double array of independent and identically distributed random variables with $Ew_{11} = 0$, $Ew_{11}^2 = 1$, and $\kappa = Ew_{11}{}^4 < \infty$. Moreover, $y_n = p/n \to y \in (0, \infty)$ as $n, p \to \infty$ and f_j is analytic on the interval including $[a(y), b(y)]$. Then the random vector $(G_n(f_1), \cdots, G_n(f_k))$ weakly converges to a k-dimensional Gaussian vector with mean vector,*

$$
\begin{aligned}
\mathrm{E}X_{f_j} &= \frac{f_j\left(a(y)\right) + f_j\left(b(y)\right)}{4} - \frac{1}{2\pi}\int_{a(y)}^{b(y)} \frac{f_j(x)}{\sqrt{4y - (x-1-y)^2}}dx \\
&\quad - \frac{(\kappa - 3)}{2\pi\mathbf{i}} \oint f_j(z) \frac{y\underline{m}^3(z)(1+\underline{m}(z))^{-3}}{1 - y\underline{m}^2(z)(1+\underline{m}(z))^{-2}}dz
\end{aligned}
$$

and covariance function

$$
\begin{aligned}
&\mathrm{Cov}\left(X_{f_j}, X_{f_\ell}\right) \\
= \quad & -\frac{1}{2\pi^2}\oint\oint \frac{f_j(z_1)f_\ell(z_2)}{(\underline{m}(z_1) - \underline{m}(z_2))^2} d\underline{m}(z_1) d\underline{m}(z_2) \\
& + y(\kappa - 3)\frac{1}{2\pi\mathbf{i}}\oint \frac{f_j(z_1)}{(1+\underline{m}(z_1))^2} d\underline{m}(z_1) \frac{1}{2\pi\mathbf{i}}\oint \frac{f_l(z_2)}{(1+\underline{m}(z_2))^2} d\underline{m}(z_2)
\end{aligned}
$$

for $j, \ell = 1, \ldots, k$ where $\underline{m}(z)$ is the Stieltjes transform of $\underline{F}^y \equiv (1-y)I_{[0,\infty)} + yF^y$ satisfying $z = -[\underline{m}(z)]^{-1} + y[1+\underline{m}(z)]^{-1}$, and the contours are non-overlapping and both contain the support set $[(1-\sqrt{y})^2 I_{(y\le 1)}, (1+\sqrt{y})^2]$ of F^y.

Lemmas S.1 is due to Theorem 1.1 of [1] and Theorem 2.1 of [12]. For a general $p \times p$ non-negative definite matrix $\mathbf{\Lambda}$, denote λ_j's the eigenvalues of $\mathbf{F\Lambda}$, $F^{y,G}$ is the limiting spectral distribution of $\mathbf{F\Lambda}$ when $p/n \to y \in (0, \infty)$, and $F^{y_{n-1},G}$ is obtained by replacing y by y_{n-1} in $F^{y,G}$ where G is the limiting spectral distribution of $\mathbf{\Lambda}$.

LEMMA S.2. *Assume that f_j is analytic in the interval including the support of $F^{y,G}$. Under Assumptions A and B and supposing that $\mathbf{\Lambda}$ has bounded spectral norm, we have that $(G_n(f_1), \ldots, G_n(f_k))$ with*

$$
G_n(f_j) = \sum_{i=1}^{p} f_j(\lambda_i) - p\int_0^{+\infty} f(x) dF^{y_{n-1},G}
$$

weakly converges to a Gaussian vector with mean $\mathrm{E}X_{f_j}$ and covariance function $\mathrm{Cov}(X_{f_j}, X_{f_\ell})$ as follows

$$
\begin{aligned}
\mathrm{E}X_{f_j} &= -\frac{1}{2\pi\mathbf{i}}\oint f_j(z)\frac{y\int \underline{m}^3(z)t^2[1+t\underline{m}(z)]^{-3}dG(t)}{\{1 - y\int \underline{m}^2(z)t^2[1+t\underline{m}(z)]^{-2}dG(t)\}^2}dz \\
&\quad - \frac{(\kappa-3)}{2\pi\mathbf{i}}\oint f_j(z)\frac{y\int \underline{m}^3(z)t^2[1+t\underline{m}(z)]^{-3}dG(t)}{1 - y\int \underline{m}^2(z)t^2[1+t\underline{m}(z)]^{-2}dG(t)}dz
\end{aligned}
$$

and

$$\begin{aligned}
&\mathrm{Cov}(X_{f_j}, X_{f_\ell})\\
= & -\frac{1}{2\pi^2}\oint\oint \frac{f_j(z_1)f_\ell(z_2)}{[\underline{m}(z_1)-\underline{m}(z_2)]^2}d\underline{m}(z_1)d\underline{m}(z_2)\\
& -\frac{y(\kappa-3)}{4\pi^2}\int\int f_j(z_1)f_\ell(z_2)\int \frac{t^2dG(t)}{[\underline{m}(z_1)t+1]^2[\underline{m}(z_2)t+1]^2}d\underline{m}(z_1)d\underline{m}(z_2),
\end{aligned}$$

where $\underline{m}(z)$ *satisfies*

$$z = -\frac{1}{\underline{m}(z)} + y\int \frac{t}{1+t\underline{m}(z)}dG(t).$$

Lemma S.2 is due to [12] and [8].

S.2. Proof of Lemma 2.1. Part (a) of Lemma 2.1 is proved from [11] and [12]. The second part of Lemma 2.1 is from Lemma S.1. We will employ Lemma S.1 to prove Part (b) of Lemma 2.1 by setting $f_1(\lambda_j) = \lambda_j$ and $f_2(\lambda) = \log(\lambda_j + \epsilon)$, where λ_j's stand for the eigenvalues of $\mathbf{F}$. By calculating the corresponding means and covariance matrix, we obtain the following lemma.

LEMMA S.3. *Under the conditions of Lemma S.1 and* $y_{n-1} \geq 1$, *it follows that for a small positive number* ϵ,

$$\begin{pmatrix} \sum_{j=1}^{n-1}\lambda_j - p \\ \sum_{j=1}^{n-1}\log\lambda_j - p\alpha_2(y_{n-1}) \end{pmatrix} \to N\left(\begin{pmatrix} m_{21}(y) \\ m_{22}(y)\end{pmatrix}, \begin{pmatrix} v_{11}(y) & v_{12}(y) \\ v_{21}(y) & v_{22}(y)\end{pmatrix}\right)$$

where $y \neq 1$,

$$\begin{aligned}
\alpha_2(y_{n-1}) &= = \int_{a(y_{n-1})}^{b(y_{n-1})} \frac{\log(x+\epsilon)}{2\pi x y_{n-1}}\sqrt{(b(y_{n-1})-x)(x-a(y_{n-1}))}dx\\
m_{21}(y) &= \frac{a(y)+b(y)}{4} - \frac{1}{2\pi}\int_{a(y)}^{b(y)} \frac{x}{\sqrt{4y-(x-1-y)^2}}dx\\
&\qquad +(\kappa-3)\cdot\frac{-1}{2\pi i}\oint z\frac{y\underline{m}^3(z)(1+\underline{m})^{-3}}{1-y\underline{m}^2(z)(1+\underline{m})^{-2}}dz
\end{aligned}$$

$$\begin{aligned} m_{22}(y) &= \frac{\log(a(y))+\log(b(y))}{4}-\frac{1}{2\pi}\int_{a(y)}^{b(y)}\frac{\log(x)}{\sqrt{4y-(x-1-y)^2}}dx \\ &\quad +(\kappa-3)\cdot\frac{-1}{2\pi i}\oint\log(z+\epsilon)\frac{y\underline{m}^3(z)(1+\underline{m})^{-3}}{1-y\underline{m}^2(z)(1+\underline{m})^{-2}}dz \\ v_{11}(y) &= -\frac{1}{2\pi^2}\oint\oint\frac{z_1z_2}{(\underline{m}(z_1)-\underline{m}(z_2))^2}d\underline{m}(z_1)d\underline{m}(z_2) \\ &\quad +y(\kappa-3)\cdot\frac{1}{2\pi i}\oint\frac{z_1}{(1+\underline{m}(z_1))^2}d\underline{m}(z_1)\cdot\frac{1}{2\pi i}\oint\frac{z_2}{(1+\underline{m}(z_2))^2}d\underline{m}(z_2) \end{aligned}$$

$$\begin{aligned} v_{12}(y) &= -\frac{1}{2\pi^2}\oint\oint\frac{z_1\log(z_2+\epsilon)}{(\underline{m}(z_1)-\underline{m}(z_2))^2}d\underline{m}(z_1)d\underline{m}(z_2) \\ &\quad +y(\kappa-3)\cdot\frac{1}{2\pi i}\oint\frac{z_1}{(1+\underline{m}(z_1))^2}d\underline{m}(z_1)\cdot\frac{1}{2\pi i}\oint\frac{\log(z_2+\epsilon)}{(1+\underline{m}(z_2))^2}d\underline{m}(z_2) \end{aligned}$$

and

$$\begin{aligned} v_{22}(y) &= -\frac{1}{2\pi^2}\oint\oint\frac{\log(z_1+\epsilon)\log(z_2+\epsilon)}{(\underline{m}(z_1)-\underline{m}(z_2))^2}d\underline{m}(z_1)d\underline{m}(z_2) \\ &\quad +y(\kappa-3)\cdot\frac{1}{2\pi i}\oint\frac{\log(z_1+\epsilon)}{(1+\underline{m}(z_1))^2}d\underline{m}(z_1)\cdot\frac{1}{2\pi i}\oint\frac{\log(z_2+\epsilon)}{(1+\underline{m}(z_2))^2}d\underline{m}(z_2) \end{aligned}$$

Now we will simplify the expressions of $\alpha_2(y)$, $m_{21}(y)$, $m_{22}(y)$, $v_{11}(y)$, $v_{12}(y)$ and $v_{22}(y)$.

Step 1. When $\lim_{n\to\infty} y_{n-1} = y > 1$, it follows that $\underline{m}(z)$ satisfies

$$z = -\frac{1}{\underline{m}(z)} + \frac{y}{1+\underline{m}(z)}$$

(Page 556 of Bai and Silverstein (2004)). Let $m_i = \underline{m}(z_i), \quad i = 1, 2$. For fixed m_2, we have on a contour enclosing $(y-1)^{-1}$, s_1 and 0,

$$\begin{aligned} &\oint\frac{\log(z(m_1)+\epsilon)}{(m_1-m_2)^2}dm_1 \\ =\ &\oint\frac{\frac{1}{m_1^2}-\frac{y}{(1+m_1)^2}}{\left(-\frac{1}{m_1}+\frac{y}{1+m_1}+\epsilon\right)}\frac{1}{(m_1-m_2)}dm_1 \\ =\ &\oint\frac{1}{[\epsilon m_1^2+(\epsilon+y-1)m_1-1](m_1-m_2)}\left(\frac{m_1+1}{m_1}-\frac{ym_1}{1+m_1}\right)dm_1 \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\epsilon}\oint \frac{1}{(m_1-s_1)(m_1-s_2)(m_1-m_2)}\left(\frac{m_1+1}{m_1}-\frac{ym_1}{1+m_1}\right)dm_1 \\
&= \frac{-2\pi i}{\epsilon}\left[\frac{s_1+1}{(s_1-s_2)s_1(m_2-s_1)}+\frac{1}{s_1s_2m_2}-\frac{ys_1}{(s_1-s_2)(s_1+1)(m_2-s_1)}\right] \\
&= \frac{-2\pi i}{\epsilon}\left[\frac{(s_1+1)^2-ys_1^2}{(s_1-s_2)s_1(s_1+1)(m_2-s_1)}+\frac{1}{s_1s_2m_2}\right]
\end{aligned}
$$

where $s_2=-\frac{1}{2}\left(1+\frac{y-1}{\epsilon}\right)-\frac{1}{2}\sqrt{\left(1+\frac{y-1}{\epsilon}\right)^2+\frac{4}{\epsilon}}$ and $s_1=-\frac{1}{2}\left(1+\frac{y-1}{\epsilon}\right)+\frac{1}{2}\sqrt{\left(1+\frac{y-1}{\epsilon}\right)^2+\frac{4}{\epsilon}}$.

$$
\oint \frac{z_1}{(m_1-m_2)^2}dm_1=\oint\frac{-\frac{1}{m_1}+\frac{y}{1+m_1}}{(m_1-m_2)^2}dm_1=\frac{-2\pi i}{m_2^2}.
$$

Then

$$
-\frac{1}{2\pi^2}\oint\oint\frac{z_1z_2}{(\underline{m}(z_1)-\underline{m}(z_2))^2}d\underline{m}(z_1)d\underline{m}(z_2)=\frac{i}{\pi}\oint\frac{1}{m_2^2}\left(-\frac{1}{m_2}+\frac{y}{1+m_2}\right)dm_2=2y,
$$

and

$$
\begin{aligned}
&-\frac{1}{2\pi^2}\oint\oint\frac{z_2\log(z_1+\epsilon)}{(\underline{m}(z_1)-\underline{m}(z_2))^2}d\underline{m}(z_1)d\underline{m}(z_2) \\
&= \frac{-1}{\epsilon\pi i}\oint\left(-\frac{1}{m_2}+\frac{y}{1+m_2}\right)\left[\frac{(s_1+1)^2-ys_1^2}{(s_1-s_2)s_1(s_1+1)(m_2-s_1)}+\frac{1}{s_1s_2m_2}\right]dm_2 \\
&= \frac{-2y}{\epsilon}\left[\frac{(s_1+1)^2-ys_1^2}{(s_1-s_2)s_1(s_1+1)^2}+\frac{1}{s_1s_2}\right].
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&-\frac{1}{2\pi^2}\oint\oint\frac{\log(z_1+\epsilon)\log(z_2+\epsilon)}{(\underline{m}(z_1)-\underline{m}(z_2))^2}d\underline{m}(z_1)d\underline{m}(z_2) \\
&= \frac{-1}{\epsilon\pi i}\oint\left[\frac{(s_1+1)^2-ys_1^2}{(s_1-s_2)s_1(s_1+1)(m_2-s_1)}+\frac{1}{s_1s_2m_2}\right]\log\left(-\frac{1}{m_2}+\frac{y}{1+m_2}+\epsilon\right)dm_2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-1}{\epsilon\pi i}\oint\left[\frac{(s_1+1)^2-ys_1^2}{(s_1-s_2)s_1(s_1+1)(m_2-s_1)}+\frac{1}{s_1s_2m_2}\right]\log\left(\epsilon\frac{m_2-s_1}{m_2}\right)dm_2\\
&\quad+\frac{-1}{\epsilon\pi i}\oint\left[\frac{(s_1+1)^2-ys_1^2}{(s_1-s_2)s_1(s_1+1)(m_2-s_1)}+\frac{1}{s_1s_2m_2}\right]\log\left(\frac{m_2-s_2}{1+m_2}\right)dm_2\\
&= -\frac{2}{\epsilon}\left[\frac{(s_1+1)^2-ys_1^2}{(s_1-s_2)s_1(s_1+1)}\log\left(\frac{s_1-s_2}{1+s_1}\right)+\frac{\log(-s_2)}{s_1s_2}\right].
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&\frac{1}{2\pi i}\oint z\frac{y\underline{m}^3(z)(1+\underline{m})^{-3}}{1-y\underline{m}^2(z)(1+\underline{m})^{-2}}dz\\
&= \frac{-y}{2\pi i}\oint\left(-\frac{1}{\underline{m}}+\frac{y}{1+\underline{m}}\right)\frac{\underline{m}}{(1+\underline{m})^3}d\underline{m}\\
&= \frac{y}{2\pi i}\oint\frac{1}{(1+\underline{m})^3}d\underline{m}-\frac{y^2}{2\pi i}\oint\frac{\underline{m}}{(1+\underline{m})^4}d\underline{m}=0
\end{aligned}
$$

where $z=-\frac{1}{\underline{m}}+\frac{y}{1+\underline{m}}$ and $\frac{d\underline{m}(z)}{dz}=\frac{\underline{m}^2}{1-y\underline{m}^2(z)(1+\underline{m})^{-2}}$ (see P596 of Bai and Silverstein (2004)).

$$
\begin{aligned}
&\frac{-1}{2\pi i}\oint\log(z+\epsilon)\frac{y\underline{m}^3(z)(1+\underline{m})^{-3}}{1-y\underline{m}^2(z)(1+\underline{m})^{-2}}dz\\
&= \frac{-y}{2\pi i}\cdot\frac{1}{2}\oint\log\left(-\frac{1}{\underline{m}}+\frac{y}{1+\underline{m}}+\epsilon\right)d\frac{\underline{m}^2}{(1+\underline{m})^2}\\
&= \frac{-y}{2\pi i}\cdot\frac{1}{2}\oint\frac{\underline{m}^2}{(1+\underline{m})^2}\frac{\frac{1}{\underline{m}^2}-\frac{y}{(1+\underline{m})^2}}{-\frac{1}{\underline{m}}+\frac{y}{1+\underline{m}}+\epsilon}d\underline{m}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-y}{2\pi\epsilon i}\cdot\frac{1}{2}\oint\frac{\underline{m}^2}{(1+\underline{m})^2(\underline{m}-s_1)(\underline{m}-s_2)}\left[\frac{1+\underline{m}}{\underline{m}}-\frac{y\underline{m}}{1+\underline{m}}\right]d\underline{m}\\
&= \frac{-y}{2\epsilon}\left[\frac{s_1}{(1+s_1)(s_1-s_2)}-\frac{ys_1^3}{(1+s_1)^3(s_1-s_2)}\right]\\
&= \frac{-y}{2\epsilon(s_1-s_2)}\left[\frac{s_1}{(1+s_1)}-\frac{ys_1^3}{(1+s_1)^3}\right].
\end{aligned}
$$

We have

$$
\begin{aligned}
\frac{1}{2\pi i}\oint\frac{z}{(1+\underline{m}(z))^2}d\underline{m}(z) &= \frac{1}{2\pi i}\oint\left(-\frac{1}{\underline{m}}+\frac{y}{1+\underline{m}}\right)\frac{1}{(1+\underline{m}(z))^2}d\underline{m}(z)\\
&= \frac{1}{2\pi i}\oint-\frac{1}{\underline{m}}\frac{1}{(1+\underline{m}(z))^2}d\underline{m}(z)+\frac{1}{2\pi i}\oint\frac{y}{(1+\underline{m})^3}d\underline{m}(z)\\
&= -1.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
m_{21}(y) &= \frac{a(y)+b(y)}{4} - \frac{1}{2\pi}\int_{a(y)}^{b(y)} \frac{x}{\sqrt{4y-(x-1-y)^2}}dx \\
m_{22}(y) &= \frac{\log(a(y))+\log(b(y))}{4} - \frac{1}{2\pi}\int_{a(y)}^{b(y)} \frac{\log(x)}{\sqrt{4y-(x-1-y)^2}}dx \\
&\quad -\frac{(\kappa-3)y}{2\epsilon(s_1-s_2)}\left[\frac{s_1}{1+s_1} - \frac{ys_1^3}{(1+s_1)^3}\right] \\
v_{11}(y) &= 2y+y(\kappa-3) \\
v_{12}(y) &= \frac{-2y}{\epsilon}\left[\frac{(s_1+1)^2-ys_1^2}{(s_1-s_2)s_1(s_1+1)^2} + \frac{1}{s_1s_2}\right] - \frac{y(\kappa-3)}{\epsilon}\left[\frac{(s_1+1)^2-ys_1^2}{(s_1-s_2)s_1(s_1+1)^2} + \frac{1}{s_1s_2}\right] \\
&= \frac{(-\kappa+1)y}{\epsilon}\left[\frac{(s_1+1)^2-ys_1^2}{(s_1-s_2)s_1(s_1+1)^2} + \frac{1}{s_1s_2}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
v_{22}(y) &= -\frac{2}{\epsilon}\left[\frac{(s_1+1)^2-ys_1^2}{(s_1-s_2)s_1(s_1+1)}\log\left(\frac{s_1-s_2}{1+s_1}\right) + \frac{\log(-s_2)}{s_1s_2}\right] \\
&\quad + \frac{y(\kappa-3)}{\epsilon^2}\left[\frac{(s_1+1)^2-ys_1^2}{(s_1-s_2)s_1(s_1+1)^2} + \frac{1}{s_1s_2}\right]^2.
\end{aligned}
$$

Step 2. We will further simplify the expressions of $\alpha_2(y)$, $m_{21}(y)$, $m_{22}(y)$, $v_{11}(y)$, $v_{12}(y)$ and $v_{22}(y)$ as $\epsilon \to 0$. We have

$$
\begin{aligned}
\alpha_2(y) &= \int_{a(y)}^{b(y)} \frac{\log(x+\epsilon)}{2\pi yx}\sqrt{(b(y)-x)(x-a(y))}dx \\
&\to \int_{a(y)}^{b(y)} \frac{\log x}{2\pi yx}\sqrt{(b(y)-x)(x-a(y))}dx \\
&= \int_0^{\pi} \frac{2(\sin\theta)^2\log(1+y-2\sqrt{y}\cos\theta)}{\pi(1+y-2\sqrt{y}\cos\theta)}d\theta \\
&= \int_0^{2\pi} \frac{(\sin\theta)^2\log(1+y-2\sqrt{y}\cos\theta)}{\pi(1+y-2\sqrt{y}\cos\theta)}d\theta \\
&= \frac{1}{-4\pi i}\int_{|z|=1} \frac{(z-1/z)^2\log[(\sqrt{y}z-1)(\sqrt{y}/z-1)]}{z(\sqrt{y}z-1)(\sqrt{y}/z-1)}dz \\
&= \frac{1}{4\pi\sqrt{y}i}\int_{|z|=1} \frac{(z^2-1)^2\log[(\sqrt{y}z-1)(\sqrt{y}/z-1)]}{z^2(z-1/\sqrt{y})(z-\sqrt{y})}dz
\end{aligned}
$$

$$
\begin{aligned}
\text{(S.1)} \quad &= \frac{1}{4\pi\sqrt{y}i}\int_{|z|=1}\log[(\sqrt{y}z-1)(\sqrt{y}/z-1)]dz\\
\text{(S.2)} \quad & \quad +\frac{1}{4\pi\sqrt{y}i}\int_{|z|=1}\frac{(\sqrt{y}+1/\sqrt{y})\log[(\sqrt{y}z-1)(\sqrt{y}/z-1)]}{z}dz\\
\text{(S.3)} \quad & \quad +\frac{1}{4\pi\sqrt{y}i}\int_{|z|=1}\frac{\log[(\sqrt{y}z-1)(\sqrt{y}/z-1)]}{z^2}dz\\
\text{(S.4)} \quad & \quad +\frac{1}{4\pi\sqrt{y}i}\int_{|z|=1}\frac{(\sqrt{y}-1/\sqrt{y})\log[(\sqrt{y}z-1)(\sqrt{y}/z-1)]}{z-\sqrt{y}}dz\\
\text{(S.5)} \quad & \quad -\frac{1}{4\pi\sqrt{y}i}\int_{|z|=1}\frac{(\sqrt{y}-1/\sqrt{y})\log[(\sqrt{y}z-1)(\sqrt{y}/z-1)]}{z-1/\sqrt{y}}dz\\
&= -y^{-1}+(1+y^{-1})\log\sqrt{y}-(1-y^{-1})\log(\sqrt{y}-1/\sqrt{y})\\
&= -y^{-1}+y^{-1}\log y-(1-y^{-1})\log(1-y^{-1})\\
&= (y^{-1}-1)\log(y-1)+\log y-y^{-1}\\
\text{(S.6)} \quad &= y^{-1}\log y+y^{-1}[(1-y)\log(1-y^{-1})-1],
\end{aligned}
$$

where (S.1)$= -0.5y^{-1}$, (S.2)$= (1+y^{-1})\log\sqrt{y}$, (S.3)$= -0.5y^{-1}$, (S.4)$= -0.5(1-y^{-1})\log(1-y^{-1})$ and (S.5)$= -0.5(1-y^{-1})\log(y-1)$.

Moreover, we have

$$\epsilon s_1 s_2 = -1, \quad \epsilon(s_1-s_2) = \sqrt{(\epsilon+y-1)^2+4\epsilon} \to y-1 \ as \ \epsilon \to 0.$$

$$\epsilon s_1 = -0.5(\epsilon+y-1)+0.5\sqrt{(\epsilon+y-1)^2+4\epsilon} \to 0 \ as \ \epsilon \to 0.$$

$$\epsilon(s_1+1) = -0.5(\epsilon+y-1)+0.5\sqrt{(\epsilon+y-1)^2+4\epsilon}+\epsilon \to 0 \ as \ \epsilon \to 0.$$

$$
\begin{aligned}
& \lim_{\epsilon\to 0}\frac{s_1}{s_1+1}\\
&= \lim_{\epsilon\to 0}\frac{-0.5(\epsilon+y-1)+0.5\sqrt{(\epsilon+y-1)^2+4\epsilon}}{-0.5(\epsilon+y-1)+0.5\sqrt{(\epsilon+y-1)^2+4\epsilon}+\epsilon}\\
&= \lim_{\epsilon\to 0}\frac{-0.5+0.5(y+1)[(\epsilon+y-1)^2+4\epsilon]^{-1/2}}{-0.5+0.5(y+1)[(\epsilon+y-1)^2+4\epsilon]^{-1/2}+1}\\
&= y^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\epsilon\to 0}\frac{1}{s_1} &= \lim_{\epsilon\to 0}\frac{\epsilon}{-0.5(\epsilon+y-1)+0.5\sqrt{(\epsilon+y-1)^2+4\epsilon}}\\
&= \lim_{\epsilon\to 0}\frac{1}{-0.5+0.5(y+1)[(\epsilon+y-1)^2+4\epsilon]^{-1/2}}\\
&= y-1
\end{aligned}
$$

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{1}{s_1+1} &= \lim_{\epsilon\to 0}\frac{\epsilon}{-0.5(\epsilon+y-1)+0.5\sqrt{(\epsilon+y-1)^2+4\epsilon}+\epsilon}\\
&= \lim_{\epsilon\to 0}\frac{1}{-0.5+0.5(y+1)[(\epsilon+y-1)^2+4\epsilon]^{-1/2}+1}\\
&= (y-1)y^{-1}
\end{aligned}$$

$$\frac{1}{\epsilon(s_1-s_2)}\left[\frac{s_1}{1+s_1}-\frac{ys_1^3}{(1+s_1)^3}\right]\to\frac{1}{y-1}(y^{-1}-y^{-2})=y^{-2}\ as\ \epsilon\to 0.$$

$$\frac{(s_1+1)^2-ys_1^2}{\epsilon(s_1-s_2)s_1(s_1+1)}\to 1\ as\ \epsilon\to 0.$$

$$\frac{(s_1+1)^2-ys_1^2}{\epsilon(s_1-s_2)s_1(s_1+1)^2}+\frac{1}{\epsilon s_1s_2}\to -y^{-1}\ as\ \epsilon\to 0.$$

$$\log\frac{\epsilon(s_1-s_2)}{1+s_1}\to\log\frac{(y-1)^2}{y}\ as\ \epsilon\to 0.$$

$$\frac{\log(-\epsilon s_2)}{\epsilon s_1s_2}\to -\log(y-1)\ as\ \epsilon\to 0.$$

$$\frac{(s_1+1)^2-ys_1^2}{\epsilon(s_1-s_2)s_1(s_1+1)}\log\frac{\epsilon(s_1-s_2)}{1+s_1}+\frac{\log(-\epsilon s_2)}{\epsilon s_1s_2}\to\log(y-1)-\log y\ as\ \epsilon\to 0.$$

$$\begin{aligned}
&\left[\frac{(s_1+1)^2-ys_1^2}{\epsilon(s_1-s_2)s_1(s_1+1)}+\frac{1}{\epsilon s_1s_2}\right]\log\epsilon\\
=\ &\frac{[(s_1+1)^2-ys_1^2]-\epsilon(s_1-s_2)s_1(s_1+1)}{\epsilon(s_1-s_2)s_1(s_1+1)}\log\epsilon\\
\to\ &0\ as\ \epsilon\to 0.
\end{aligned}$$

Then we have

$$\begin{aligned}
m_{21}(y) &= \frac{a(y)+b(y)}{4}-\frac{1}{2\pi}\int_{a(y)}^{b(y)}\frac{x}{\sqrt{4y-(x-1-y)^2}}dx\\
&= \frac{a(y)+b(y)}{4}-\frac{1}{2\pi}\int_0^{\pi}(1+y-2\sqrt{y}\cos\theta)d\theta\\
&= (1+y)/2-\frac{1+y}{2}=0
\end{aligned}$$

$$\begin{aligned}
m_{22}(y) &= \frac{\log a(y)+\log b(y)}{4} - \frac{1}{2\pi}\int_{a(y)}^{b(y)} \frac{\log x}{\sqrt{4y-(x-1-y)^2}}dx \\
&\quad -\frac{(\kappa-3)y}{2\epsilon(s_1-s_2)}\left[\frac{s_1}{1+s_1}-\frac{ys_1^3}{(1+s_1)^3}\right] \\
&\to \frac{\log(y-1)}{2} - \frac{1}{2\pi}\int_0^{\pi}\log(1+y-2\sqrt{y}\cos\theta)d\theta - \frac{(\kappa-3)}{2y} \\
&= \frac{\log(y-1)}{2} - \frac{1}{4\pi}\int_0^{2\pi}\log(1+y-2\sqrt{y}\cos\theta)d\theta - \frac{(\kappa-3)}{2y} \\
&= \frac{\log(y-1)}{2} - \frac{1}{4\pi i}\oint_{|z|=1}\frac{\log[1+y-\sqrt{y}(z+z^{-1})]}{z}dz - \frac{(\kappa-3)}{2y} \\
&= \frac{\log(y-1)}{2} - \log(\sqrt{y}) - \frac{(\kappa-3)}{2y},
\end{aligned}$$

where

$$\frac{1}{4\pi i}\oint_{|z|=1}\frac{\log[1+y-\sqrt{y}(z+z^{-1})]}{z}dz = \log(\sqrt{y}).$$

Thus we have

$$\alpha_2(y) = (y^{-1}-1)\log(y-1)+\log y - y^{-1}$$

$$m_{21}(y)=0, \quad m_{22}(y) = 0.5\log(1-y^{-1}) - 0.5(\kappa-3)y^{-1}$$

and

$$\nu_{11}(y) = y(\kappa-1), \quad \nu_{12}(y)=\nu_{21}(y)=\kappa-1, \quad \nu_{22}(y) = -2\log(1-y^{-1})+(\kappa-3)y^{-1}.$$

Then Part (b) of Lemma 2.1 is obtained.

S.3. Proof of Theorem 2.4 (a) and (b). Under H_1, $\hat{\theta}_k = \theta_k^* + O_p(1/n)$, $k=1,\cdots,K$. Thus, it follows by Taylor's expansion that

$$\begin{aligned}
&\mathrm{tr}\mathbf{S}_n\widehat{\boldsymbol{\Sigma}}_0^{-1} - \log|\mathbf{S}_n\widehat{\boldsymbol{\Sigma}}_0^{-1}| \\
=\;& \mathrm{tr}\mathbf{S}_n\boldsymbol{\Sigma}_{0*}^{-1} - \sum_{k=1}^K p(\hat{\theta}_k-\theta_k^*)p^{-1}\mathrm{tr}\mathbf{S}_n\boldsymbol{\Sigma}_1^{*-1}\mathbf{A}_k\boldsymbol{\Sigma}_1^{*-1} \\
&-\log|\mathbf{S}_n\boldsymbol{\Sigma}_1^{*-1}| + \sum_{k=1}^K p(\hat{\theta}_k-\theta_k^*)p^{-1}\mathrm{tr}\mathbf{A}_k\boldsymbol{\Sigma}_1^{*-1} + o_p(1)
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{tr}\mathbf{S}_n\mathbf{\Sigma}_1^{*-1} + \sum_{k=1}^{K} p(\hat{\theta}_k - \theta_k^*)p^{-1}[\mathrm{tr}\mathbf{A}_k\mathbf{\Sigma}_1^{*-1} - \mathrm{tr}\mathbf{\Sigma}\mathbf{\Sigma}_1^{*-1}\mathbf{A}_k\mathbf{\Sigma}_1^{*-1}] \\
&\quad - \log|\mathbf{S}_n\mathbf{\Sigma}_1^{*-1}| + o_p(1) \\
&= \mathrm{tr}\mathbf{S}_n\mathbf{B}_* - \log|\mathbf{S}_n\mathbf{\Sigma}_1^{*-1}| + \mathrm{tr}\mathbf{\Sigma}\mathbf{\Sigma}_1^{*-1} - p + o_p(1) \\
&= \mathrm{tr}\mathbf{F}\mathbf{\Gamma}^T\mathbf{B}_*\mathbf{\Gamma} - \log|\mathbf{F}\mathbf{\Gamma}^T\mathbf{B}_*\mathbf{\Gamma}| + \log|\mathbf{B}_*\mathbf{\Sigma}_1^*| + \mathrm{tr}\mathbf{\Sigma}\mathbf{\Sigma}_1^{*-1} - p + o_p(1).
\end{aligned}
$$

The last equality holds because $p^{-1}\mathrm{tr}\mathbf{S}_n\mathbf{\Sigma}_1^{*-1}\mathbf{A}_k\mathbf{\Sigma}_1^{*-1} - p^{-1}\mathrm{tr}\mathbf{\Sigma}\mathbf{\Sigma}_1^{*-1}\mathbf{A}_k\mathbf{\Sigma}_1^{*-1} = o_p(1)$ by (2.10) and defining $\mathbf{B}_* = \mathbf{\Sigma}_1^{*-1} + \sum_{k=1}^{K} h_k\mathbf{A}_k$ with $(h_1, \ldots, h_K) = (\mathrm{tr}\mathbf{A}_1\mathbf{E}, \ldots, \mathrm{tr}\mathbf{A}_K\mathbf{E})\mathbf{D}$ and $\mathbf{E} = \mathbf{\Sigma}_1^{*-1} - \mathbf{\Sigma}_1^{*-1}\mathbf{\Sigma}\mathbf{\Sigma}_1^{*-1}$. By Lemma S.2 and when $p < n-1$, we have

$$
\frac{\mathrm{tr}\mathbf{S}_n\widehat{\mathbf{\Sigma}}_0^{-1} - \log|\mathbf{S}_n\widehat{\mathbf{\Sigma}}_0^{-1}| - pF_1^{y_{n-1},G} - \mu_1^{(1)}}{\sigma_{n1}^{(1)}} \to N(0,1),
$$

where $F_1^{y_{n-1},G} = \int(x - \log x - 1)f^{y_{n-1},G}(x)dx$ with the density $f^{y_{n-1},G}(x) = (\pi y_{n-1})^{-1}\lim_{z\to x}\Im(\underline{m}(z))$, $z = -[\underline{m}(z)]^{-1} + y\int t[1 + t\underline{m}(z)]^{-1}dG(t)$,

$$
\begin{aligned}
\mu_1^{(1)} &= \log|\mathbf{B}_*\mathbf{\Sigma}_1^*| + \mathrm{tr}\mathbf{\Sigma}\mathbf{\Sigma}_1^{*-1} - p \\
&\quad - \frac{1}{2\pi\mathbf{i}}\oint(z - \log z)\frac{y\int \underline{m}^3(z)t^2(1+t\underline{m}(z))^{-3}dG(t)}{(1 - y\int \underline{m}^2(z)t^2(1+t\underline{m}(z))^{-2}dG(t))^2}dz \\
&\quad - \frac{(\kappa-3)}{2\pi\mathbf{i}}\oint(z - \log z)\frac{y\int \underline{m}^3(z)t^2(1+t\underline{m}(z))^{-3}dG(t)}{1 - y\int \underline{m}^2(z)t^2(1+t\underline{m}(z))^{-2}dG(t)}dz
\end{aligned}
\tag{S.1}
$$

$$
\begin{aligned}
&(\sigma_{n1}^{(1)})^2 \\
&= -\frac{1}{2\pi^2}\oint\oint\frac{(z_1 - \log z_1)(z_2 - \log z_2)}{(\underline{m}(z_1) - \underline{m}(z_2))^2}d\underline{m}(z_1)d\underline{m}(z_2) \\
&\quad - \frac{y(\kappa-3)}{4\pi^2}\int\int\left(\int\frac{(z_1 - \log z_1)(z_2 - \log z_2)t^2}{(\underline{m}(z_1)t+1)^2(\underline{m}(z_2)t+1)^2}dG(t)\right)d\underline{m}(z_1)d\underline{m}(z_2)
\end{aligned}
$$

and G is the limiting spectral distribution of $\mathbf{\Lambda} = \mathbf{\Gamma}^T\mathbf{B}_*\mathbf{\Gamma}$.

When $p \geq n-1$ and $y \neq 1$, the test statistic is as follows

$$
\frac{n-1}{p}\sum_{j=1}^{n-1}\lambda_j - \sum_{j=1}^{n-1}\log\lambda_j
$$

where $\{\lambda_1 \geq \ldots \geq \lambda_p\}$ are the eigenvalues of $\mathbf{S}_n\widehat{\mathbf{\Sigma}}_0^{-1}$. By Lemma S.2 and when $p \geq n-1$, we have

$$\frac{(y_{n-1})^{-1}\sum_{j=1}^{n-1}\lambda_j - \sum_{j=1}^{n-1}\log\lambda_j - pF_2^{y_{n-1},G} - \mu_2^{(1)}}{\sigma_{n2}^{(1)}} \to N(0,1)$$

where $F_2^{y_{n-1},G} = \int[(y_{n-1})^{-1}x - \log x - 1]f^{y_{n-1},G}(x)dx$,

$$\begin{aligned}\mu_2^{(1)} &= \log|\mathbf{B}_*\mathbf{\Sigma}_1^*| + y_{n-1}^{-1}\mathrm{tr}\mathbf{\Sigma}\mathbf{\Sigma}_1^{*-1} - p \\ &-\frac{1}{2\pi\mathbf{i}}\oint[(y_{n-1})^{-1}z - \log z]\frac{y\int \underline{m}^3(z)t^2(1+t\underline{m}(z))^{-3}dG(t)}{(1-y\int \underline{m}^2(z)t^2(1+t\underline{m}(z))^{-2}dG(t))^2}dz \\ &-\frac{(\kappa-3)}{2\pi\mathbf{i}}\oint[(y_{n-1})^{-1}z - \log z]\frac{y\int \underline{m}^3(z)t^2(1+t\underline{m}(z))^{-3}dG(t)}{1-y\int \underline{m}^2(z)t^2(1+t\underline{m}(z))^{-2}dG(t)}dz,\end{aligned}$$

$$\begin{aligned}(\sigma_{n2}^{(1)})^2 &= -\frac{1}{2\pi^2}\oint\oint\frac{[(y)^{-1}z_1 - \log z_1][(y)^{-1}z_2 - \log z_2]}{(\underline{m}(z_1)-\underline{m}(z_2))^2}d\underline{m}(z_1)d\underline{m}(z_2) \\ &-\frac{y(\kappa-3)}{4\pi^2}\int\int\int\frac{(\frac{z_1}{y} - \log z_1)(\frac{z_2}{y} - \log z_2)t^2}{[\underline{m}(z_1)t+1]^2[\underline{m}(z_2)t+1]^2}dG(t)d\underline{m}(z_1)d\underline{m}(z_2)\end{aligned}$$

and G is the limiting spectral distribution of $\mathbf{\Lambda} = \mathbf{\Gamma}^T\mathbf{B}_*\mathbf{\Gamma}$.

S.4. Proof of Theorem 2.5. If $\mathbf{\Gamma}^T\mathbf{\Sigma}_1^{*-1}\mathbf{\Gamma} = \mathbf{I}_p + \mathbf{A}$ where $\mathbf{A} \geq \mathbf{0}$, it follows by direct calculations that

$$\begin{aligned}\mu_3^{(1)} &= py_{n-1} + (\kappa-2)y_{n-1} + (\mathrm{tr}\mathbf{A})^2/(n-1) + \frac{n}{n-1}\mathrm{tr}\mathbf{A}^2 + \frac{2(p+1)}{n-1}\mathrm{tr}\mathbf{A} \\ &+\frac{\kappa-3}{n-1}\sum_{i=1}^{p}(\mathbf{e}_i^T\mathbf{A}\mathbf{e}_i)^2 + \frac{2(\kappa-3)}{n-1}\mathbf{trA}. \qquad \text{(S.1)}\end{aligned}$$

Note that the leading term in the asymptotical mean of T_{n2} under H_0 is $\mu_0 = (p+\kappa-2)y_{n-1}$. If $\kappa - 3 \geq 0$, it follows that

$$\mu_3^{(1)} - (p+\kappa-2)y_{n-1} \geq \frac{n}{n-1}\mathrm{tr}\mathbf{A}^2 > \delta$$

by the assumption $\mathrm{tr}\mathbf{A}^2 > \delta > 0$. On the other hand, if $\kappa - 3 < 0$

$$\begin{aligned}
&\mu_3^{(1)} - (p+\kappa-2)y_{n-1} \\
\geq\ & (\mathrm{tr}\mathbf{A})^2/(n-1) + \frac{n}{n-1}\mathrm{tr}\mathbf{A}^2 + \frac{2(p+1)}{n-1}\mathrm{tr}\mathbf{A} \\
&+\frac{\kappa-3}{n-1}\mathrm{tr}\mathbf{A}^2 + \frac{2(\kappa-3)}{n-1}\mathbf{trA} \\
=\ & (\mathrm{tr}\mathbf{A})^2/(n-1) + \frac{n+\kappa-3}{n-1}\mathrm{tr}\mathbf{A}^2 + \frac{2(p+\kappa-2)}{n-1}\mathrm{tr}\mathbf{A} \qquad \text{(S.2)}
\end{aligned}$$

since $\sum_{i=1}^p (\mathbf{e}_i^T\mathbf{A}\mathbf{e}_i)^2 \leq \mathrm{tr}\mathbf{A}^2$. Thus, for n and p large enough $\mu_3^{(1)} - \{py_{n-1} + (\kappa-2)y_{n-1}\} \geq \delta$ under assumption $\mathrm{tr}\mathbf{A}^2 > \delta > 0$ because κ is finite. Hence it follows that $\mu_3^{(1)} > py_{n-1} + (\kappa-2)y_{n-1} + \delta$. Furthermore, it can be shown that $(\sigma_{n3}^{(1)})^2 - 4\sigma^2 \to 0$ under conditions of Theorem 2.5. Then when n is large enough, $\beta_{T_{n2}} > \alpha$. This shows the first part of the theorem.

By (S.1) and $p^{-1}(\mathrm{tr}\mathbf{A})^2 \leq \sum_{i=1}^p (\mathbf{e}_i^T\mathbf{A}\mathbf{e}_i)^2 \leq \mathrm{tr}\mathbf{A}^2$, it follows that if $\kappa \geq 3$,

$$\begin{aligned}
&\mu_3^{(1)} - (p+\kappa-2)y_{n-1} \\
\geq\ & (\mathrm{tr}\mathbf{A})^2/(n-1) + \frac{n}{n-1}\mathrm{tr}\mathbf{A}^2 + \frac{2(p+1)}{n-1}\mathrm{tr}\mathbf{A} \\
&+\frac{\kappa-3}{n-1}p^{-1}(\mathrm{tr}\mathbf{A})^2 + \frac{2(\kappa-3)}{n-1}\mathbf{trA} \\
\geq\ & (\mathrm{tr}\mathbf{A})^2/(n-1) + \frac{\kappa-3}{p(n-1)}(\mathrm{tr}\mathbf{A})^2 + \frac{n}{n-1}p^{-1}(\mathrm{tr}\mathbf{A})^2 + \frac{2(p+\kappa-2)}{n-1}\mathrm{tr}\mathbf{A} \\
=\ & (p + (\kappa-3) + py_n^{-1})y_{n-1}(p^{-1}\mathrm{tr}\mathbf{A})^2 + 2(p+\kappa-2)y_{n-1}(p^{-1}\mathrm{tr}\mathbf{A}).
\end{aligned}$$

Then

$$p^{-1}\{\mu_3^{(1)} - (p+\kappa-2)y_{n-1}\} \geq (1+y)c_1^2 + 2yc_1 \qquad \text{(S.3)}$$

since $p^{-1}\mathrm{tr}\mathbf{A} \to c_1 > 0$ and $y_{n-1} \to y$. On the other hand, if $\kappa < 3$, it follows by (S.2) that

$$\begin{aligned}
&p^{-1}\{\mu_3^{(1)} - (p+\kappa-2)y_{n-1}\} \\
\geq\ & y_{n-1}(p^{-1}\mathrm{tr}\mathbf{A})^2 + \frac{n+\kappa-3}{n-1}p^{-1}\mathrm{tr}\mathbf{A}^2 + \frac{2(p+\kappa-2)}{n-1}(p^{-1}\mathrm{tr}\mathbf{A}) \\
\geq\ & (y_{n-1} + \frac{n+\kappa-3}{n-1})(p^{-1}\mathrm{tr}\mathbf{A})^2 + \frac{2(p+\kappa-2)}{n-1}(p^{-1}\mathrm{tr}\mathbf{A}) \\
\to\ & (y+1)c_1^2 + 2yc_1 \qquad \text{(S.4)}
\end{aligned}$$

Table S.1
Performance of three estimators of κ for normal distribution under H_0. The true value of $\kappa = 3$.

n		$p = 50$ mean (stdErr)	$p = 100$ mean (stdErr)	$p = 500$ mean (stdErr)	$p = 1000$ mean (stdErr)
		Example 3.1 with $\theta_3 = 0$			
100	$\hat{\kappa}_0$	2.9530 (0.0818)	2.9513 (0.0607)	2.9480 (0.0311)	2.9473 (0.0240)
	$\hat{\kappa}_1$	2.9805 (0.7481)	3.0010 (1.1747)	3.1465 (5.0753)	3.1891 (9.8724)
	$\hat{\kappa}_2$	2.9793 (0.7631)	3.0000 (1.1979)	3.1444 (5.1753)	3.1867 (10.0681)
200	$\hat{\kappa}_0$	2.9767 (0.0576)	2.9758 (0.0427)	2.9742 (0.0218)	2.9743 (0.0170)
	$\hat{\kappa}_1$	3.0041 (0.5287)	3.0072 (0.9138)	3.0619 (3.6808)	3.0123 (7.0418)
	$\hat{\kappa}_2$	3.0040 (0.5339)	3.0070 (0.9229)	3.0613 (3.7173)	3.0101 (7.1114)
		Example 3.2 with $\theta_4 = 0$			
100	$\hat{\kappa}_0$	2.9580 (0.0725)	2.9580 (0.0513)	2.9578 (0.0230)	2.9579 (0.0162)
	$\hat{\kappa}_1$	2.9901 (0.3124)	2.9916 (0.3038)	3.0087 (0.2977)	2.9993 (0.2904)
	$\hat{\kappa}_2$	2.9894 (0.3186)	2.9910 (0.3098)	3.0085 (0.3037)	2.9988 (0.2961)
200	$\hat{\kappa}_0$	2.9792 (0.0509)	2.9791 (0.0361)	2.9791 (0.0161)	2.9792 (0.0114)
	$\hat{\kappa}_1$	2.9985 (0.2138)	2.9981 (0.2166)	2.9952 (0.2079)	3.0018 (0.2048)
	$\hat{\kappa}_2$	2.9984 (0.2159)	2.9980 (0.2188)	2.9950 (0.2100)	3.0017 (0.2069)
		Example 3.3: Factor model with $\theta_4 = 0$			
100	$\hat{\kappa}_0$	2.9063 (0.2560)	2.8827 (0.3060)	2.8180 (0.4985)	2.7931 (0.5977)
	$\hat{\kappa}_1$	3.0807 (2.2335)	2.9935 (0.3039)	3.0123 (0.2978)	3.0001 (0.2948)
	$\hat{\kappa}_2$	3.0800 (2.2779)	2.9930 (0.3099)	3.0121 (0.3038)	2.9997 (0.3007)
200	$\hat{\kappa}_0$	2.9536 (0.1850)	2.9415 (0.2199)	2.9093 (0.3551)	2.8955 (0.4154)
	$\hat{\kappa}_1$	3.0136 (2.0583)	3.0100 (0.2169)	2.9963 (0.2145)	2.9915 (0.2108)
	$\hat{\kappa}_2$	3.0129 (2.0788)	3.0099 (0.2190)	2.9962 (0.2166)	2.9913 (0.2129)
		Example 3.4: Pattern with $\theta_4 = 0$			
100	$\hat{\kappa}_0$	2.9590 (0.0706)	2.9589 (0.0501)	2.9586 (0.0226)	2.9585 (0.0161)
	$\hat{\kappa}_1$	3.0086 (0.2870)	2.9943 (0.3039)	2.9885 (0.2933)	2.9807 (0.2956)
	$\hat{\kappa}_2$	3.0083 (0.2927)	2.9938 (0.3099)	2.9878 (0.2992)	2.9798 (0.3015)
200	$\hat{\kappa}_0$	2.9797 (0.0496)	2.9796 (0.0348)	2.9795 (0.0158)	2.9794 (0.0113)
	$\hat{\kappa}_1$	3.0064 (0.2054)	2.9946 (0.2165)	3.0057 (0.2180)	3.0024 (0.2127)
	$\hat{\kappa}_2$	3.0063 (0.2075)	2.9944 (0.2187)	3.0057 (0.2202)	3.0023 (0.2148)

Thus, it follows by (S.3) and (S.4) that for n large enough,

$$p^{-1}\{\mu_3^{(1)} - (p + \kappa - 2)y_{n-1}\} \geq (1 + y)c_1^2 + 2yc_1 \tag{S.5}$$

If $p^{-1}\text{tr}\mathbf{A} \to c_1 \neq 0$, then $\mu_3^{(1)} - (py_{n-1} + (\kappa - 2)y_{n-1}) \to +\infty$ by (S.1) and $(\sigma_{n3}^{(1)})^2 \to c_2 > 0$. Then we have $\beta_{T_{n2}} \to 1$. This proves the second part of the theorem.

S.5. Additional numerical study.

S.5.1. *Performance of estimators of κ.* This section presents simulation results to demonstrate the performance of three estimators of κ proposed in Section 3.1 under the settings of Examples 3.1—3.4 under both null and alternative hypotheses. Tables S.1 and S.2 present the simulation results under null hypotheses. Tables S.3 and S.4 display the simulation results under alternative hypotheses. From Tables S.1—S.4, it seems that $\hat{\kappa}_0$ performs the best across all these examples. Thus, our numerical results in Section 3 is based on $\hat{\kappa}_0$.

TABLE S.2
Performance of three estimators of κ for Gamma distribution under H_0. The true value of $\kappa = 4.50$.

		$p = 50$ mean (stdErr)	$p = 100$ mean (stdErr)	$p = 500$ mean (stdErr)	$p = 1000$ mean (stdErr)
		Example 3.1 with $\theta_3 = 0$			
100	$\hat{\kappa}_0$	4.4133 (0.3435)	4.4162 (0.2469)	4.4165 (0.1132)	4.4171 (0.0809)
	$\hat{\kappa}_1$	4.3247 (1.1438)	4.4223 (1.7019)	4.5576 (6.0840)	4.1080 (10.1517)
	$\hat{\kappa}_2$	4.3510 (1.1641)	4.4498 (1.7321)	4.5839 (6.1998)	4.1197 (10.3445)
200	$\hat{\kappa}_0$	4.4567 (0.2461)	4.4580 (0.1764)	4.4590 (0.0809)	4.4593 (0.0571)
	$\hat{\kappa}_1$	4.4697 (0.9163)	4.4596 (1.2255)	4.3632 (3.8711)	4.0541 (7.1540)
	$\hat{\kappa}_2$	4.4843 (0.9245)	4.4740 (1.2364)	4.3760 (3.9078)	4.0618 (7.2217)
		Example 3.2 with $\theta_4 = 0$			
100	$\hat{\kappa}_0$	4.4206 (0.3336)	4.4230 (0.2371)	4.4371 (0.1066)	4.4274 (0.0755)
	$\hat{\kappa}_1$	4.3777 (0.5799)	4.3883 (0.5325)	4.4152 (0.5071)	4.4168 (0.4842)
	$\hat{\kappa}_2$	4.4049 (0.5913)	4.4156 (0.5430)	4.4431 (0.5171)	4.4447 (0.4939)
200	$\hat{\kappa}_0$	4.4604 (0.2388)	4.4621 (0.1692)	4.4635 (0.0758)	4.4640 (0.0536)
	$\hat{\kappa}_1$	4.4702 (0.4312)	4.4561 (0.3811)	4.4634 (0.3566)	4.4642 (0.3628)
	$\hat{\kappa}_2$	4.4848 (0.4355)	4.4705 (0.3848)	4.4779 (0.3602)	4.4788 (0.3665)
		Example 3.3: Factor model with $\theta_4 = 0$			
100	$\hat{\kappa}_0$	4.3623 (0.4927)	4.3457 (0.4537)	4.2883 (0.5341)	4.2654 (0.6287)
	$\hat{\kappa}_1$	4.4520 (2.5029)	4.4504 (0.5294)	4.4319 (0.5380)	4.4158 (0.5147)
	$\hat{\kappa}_2$	4.4790 (2.5527)	4.4790 (0.5398)	4.4601 (0.5487)	4.4436 (0.5251)
200	$\hat{\kappa}_0$	4.4317 (0.3537)	4.4234 (0.3256)	4.3933 (0.3806)	4.3821 (0.4285)
	$\hat{\kappa}_1$	4.4919 (1.6169)	4.4457 (0.3965)	4.4549 (0.3628)	4.4623 (0.3613)
	$\hat{\kappa}_2$	4.5064 (1.6330)	4.4601 (0.4005)	4.4693 (0.3664)	4.4768 (0.3649)
		Example 3.4: Pattern with $\theta_4 = 0$			
100	$\hat{\kappa}_0$	4.4213 (0.3306)	4.4251 (0.2356)	4.4276 (0.1061)	4.4280 (0.0750)
	$\hat{\kappa}_1$	4.4086 (0.5693)	4.4073 (0.5429)	4.4087 (0.4819)	4.4356 (0.5162)
	$\hat{\kappa}_2$	4.4364 (0.5805)	4.4350 (0.5536)	4.4363 (0.4915)	4.4639 (0.5265)
200	$\hat{\kappa}_0$	4.4609 (0.2368)	4.4627 (0.1683)	4.4640 (0.0755)	4.4622 (0.0529)
	$\hat{\kappa}_1$	4.4360 (0.4137)	4.4476 (0.4006)	4.4530 (0.3554)	4.4491 (0.3414)
	$\hat{\kappa}_2$	4.4502 (0.4177)	4.4620 (0.4046)	4.4674 (0.3590)	4.4635 (0.3448)

Table S.3
Performance of three estimators of κ for normal distribution alternative hypotheses. The true value of $\kappa = 3$.

n		$p = 50$ mean (stdErr)	$p = 100$ mean (stdErr)	$p = 500$ mean (stdErr)	$p = 1000$ mean (stdErr)
		Example 3.1 with $\theta_3 = 1$			
100	$\hat{\kappa}_0$	2.9557 (0.0788)	2.9526 (0.0585)	2.9495 (0.0307)	2.9493 (0.0245)
	$\hat{\kappa}_1$	3.0079 (0.7330)	2.9925 (1.1563)	3.0520 (4.9173)	3.1955 (9.3403)
	$\hat{\kappa}_2$	3.0072 (0.7475)	2.9912 (1.1790)	3.0484 (5.0145)	3.1907 (9.5233)
200	$\hat{\kappa}_0$	2.9792 (0.0572)	2.9757 (0.0421)	2.9749 (0.0220)	2.9747 (0.0177)
	$\hat{\kappa}_1$	3.0171 (0.5344)	3.0189 (0.8504)	2.9719 (3.4022)	3.0755 (6.9724)
	$\hat{\kappa}_2$	3.0171 (0.5397)	3.0188 (0.8588)	2.9705 (3.4360)	3.0740 (7.0414)
		Example 3.2 with $\theta_4 = 1$			
100	$\hat{\kappa}_0$	2.9614 (0.0700)	2.9590 (0.0498)	2.9587 (0.0223)	2.9587 (0.0157)
	$\hat{\kappa}_1$	3.0175 (0.3790)	3.0052 (0.3606)	2.9944 (0.7543)	2.9871 (1.5296)
	$\hat{\kappa}_2$	3.0174 (0.3866)	3.0048 (0.3678)	2.9934 (0.7692)	2.9853 (1.5598)
200	$\hat{\kappa}_0$	2.9812 (0.0508)	2.9786 (0.0359)	2.9793 (0.0161)	2.9796 (0.0115)
	$\hat{\kappa}_1$	3.0096 (0.2721)	3.0007 (0.3292)	2.9973 (0.6526)	3.0192 (1.0831)
	$\hat{\kappa}_2$	3.0096 (0.2748)	3.0005 (0.3325)	2.9970 (0.6592)	3.0190 (1.0939)
		Example 3.3: Factor model with $\theta_4 = 1$			
100	$\hat{\kappa}_0$	2.9060 (0.3036)	2.8962 (0.2488)	2.8404 (0.3611)	2.8444 (0.3818)
	$\hat{\kappa}_1$	3.0490 (2.7991)	2.9295 (3.9489)	2.2102 (18.8417)	2.8875 (35.9035)
	$\hat{\kappa}_2$	3.0471 (2.8545)	2.9238 (4.0275)	2.1745 (19.2136)	2.8487 (36.6169)
200	$\hat{\kappa}_0$	2.9536 (0.2211)	2.9476 (0.1829)	2.9249 (0.2745)	2.9277 (0.2659)
	$\hat{\kappa}_1$	3.0203 (2.0463)	3.0162 (3.0206)	3.1179 (13.1557)	3.6323 (25.2584)
	$\hat{\kappa}_2$	3.0199 (2.0668)	3.0151 (3.0507)	3.1139 (13.2865)	3.6293 (25.5089)
		Example 3.4: Pattern with $\theta_4 = 1$			
100	$\hat{\kappa}_0$	2.9559 (0.1213)	2.9595 (0.0560)	2.9577 (0.0372)	2.9588 (0.0300)
	$\hat{\kappa}_1$	3.0237 (1.2048)	3.0159 (0.7985)	2.9242 (4.6263)	3.0857 (8.6986)
	$\hat{\kappa}_2$	3.0230 (1.2286)	3.0153 (0.8143)	2.9190 (4.7183)	3.0804 (8.8708)
200	$\hat{\kappa}_0$	2.9801 (0.0895)	2.9785 (0.0428)	2.9789 (0.0252)	2.9793 (0.0225)
	$\hat{\kappa}_1$	3.0148 (0.8788)	3.0009 (0.9533)	2.9389 (3.5745)	3.0680 (5.9814)
	$\hat{\kappa}_2$	3.0146 (0.8875)	3.0006 (0.9628)	2.9372 (3.6102)	3.0670 (6.0407)

TABLE S.4
Performance of three estimators of κ for Gamma distribution under alternative hypotheses. The true value of $\kappa = 4.50$.

		$p = 50$ mean (stdErr)	$p = 100$ mean (stdErr)	$p = 500$ mean (stdErr)	$p = 1000$ mean (stdErr)
		Example 3.1 with $\theta_3 = 1$			
100	$\hat{\kappa}_0$	4.3092 (0.3265)	4.3134 (0.2352)	4.2625 (0.1045)	4.2538 (0.0761)
	$\hat{\kappa}_1$	4.3978 (1.2567)	4.4088 (1.6472)	4.4833 (5.3747)	4.4649 (9.7209)
	$\hat{\kappa}_2$	4.4253 (1.2794)	4.4361 (1.6770)	4.5094 (5.4773)	4.4867 (9.9098)
200	$\hat{\kappa}_0$	4.3760 (0.2404)	4.3689 (0.1675)	4.3340 (0.0756)	4.3231 (0.0550)
	$\hat{\kappa}_1$	4.4455 (0.9009)	4.4303 (1.1921)	4.5244 (3.8139)	4.3396 (7.1114)
	$\hat{\kappa}_2$	4.4599 (0.9089)	4.4444 (1.2026)	4.5385 (3.8501)	4.3511 (7.1804)
		Example 3.2 with $\theta_4 = 1$			
100	$\hat{\kappa}_0$	4.3070 (0.3146)	4.3137 (0.2235)	4.2667 (0.0974)	4.2588 (0.0696)
	$\hat{\kappa}_1$	4.4154 (0.6827)	4.4125 (0.6121)	4.4018 (1.0712)	4.3918 (1.6428)
	$\hat{\kappa}_2$	4.4432 (0.6961)	4.4403 (0.6243)	4.4287 (1.0924)	4.4180 (1.6752)
200	$\hat{\kappa}_0$	4.3698 (0.2316)	4.3667 (0.1619)	4.3317 (0.0704)	4.3235 (0.0503)
	$\hat{\kappa}_1$	4.4573 (0.4921)	4.4564 (0.4657)	4.4696 (0.7305)	4.4529 (1.1070)
	$\hat{\kappa}_2$	4.4718 (0.4970)	4.4709 (0.4703)	4.4841 (0.7377)	4.4670 (1.1180)
		Example 3.3: Factor model with $\theta_4 = 1$			
100	$\hat{\kappa}_0$	4.1429 (0.5132)	4.0556 (0.3274)	3.8574 (0.3998)	3.8414 (0.4032)
	$\hat{\kappa}_1$	4.3818 (3.3230)	4.4068 (4.0766)	4.5150 (17.9514)	4.6174 (36.3553)
	$\hat{\kappa}_2$	4.4068 (3.3885)	4.4316 (4.1572)	4.5258 (18.3078)	4.6084 (37.0759)
200	$\hat{\kappa}_0$	4.2356 (0.3728)	4.1176 (0.2468)	3.9966 (0.2750)	3.9377 (0.3119)
	$\hat{\kappa}_1$	4.3933 (2.4653)	4.5026 (2.9993)	4.6749 (13.7814)	4.6767 (27.6974)
	$\hat{\kappa}_2$	4.4065 (2.4896)	4.5166 (3.0291)	4.6865 (13.9184)	4.6825 (27.9726)
		Example 3.3: Pattern with $\theta_4 = 1$			
100	$\hat{\kappa}_0$	4.0996 (0.3412)	4.2006 (0.2162)	4.0663 (0.1002)	4.0517 (0.0767)
	$\hat{\kappa}_1$	4.3999 (1.5883)	4.4046 (1.0778)	4.4309 (5.1010)	4.1312 (9.0821)
	$\hat{\kappa}_2$	4.4268 (1.6197)	4.4319 (1.0992)	4.4555 (5.2015)	4.1463 (9.2615)
200	$\hat{\kappa}_0$	4.1624 (0.2569)	4.2630 (0.1562)	4.1250 (0.0710)	4.1241 (0.0550)
	$\hat{\kappa}_1$	4.4568 (1.2040)	4.4543 (0.7965)	4.4369 (3.4868)	4.4604 (5.9975)
	$\hat{\kappa}_2$	4.4711 (1.2159)	4.4686 (0.8044)	4.4503 (3.5215)	4.4732 (6.0572)

TABLE S.5
Empirical sizes (in percentage) at level $\alpha = 0.05$

n	Method	$p = 270$	385	500	625	670	785	900	1025
					$\nu = 0$				
60	ZLST	6.3	4.9	4.1	4.6	5.4	6.6	6.3	6.4
	QL	5.7	4.3	5.5	4.2	4.9	5.9	5.5	4.8
	EL	5.3	5.1	5.5	4.7	6.4	6.3	7.6	7.9
	SR	5.6	4.7	5.2	4.2	5.9	6.1	4.7	4.4
80	ZLST	5.0	4.9	4.3	5.3	4.5	5.6	5.8	5
	QL	6.1	5.2	4.9	5.0	6.4	6.7	5.2	4.4
	EL	5.3	5.9	4.8	5.3	6.9	6.7	5.8	5.1
	SR	4.3	4.5	5.0	4.8	4.5	4.5	5.8	4.2
100	ZLST	5.1	5.3	6.4	4.6	5.3	6.5	4	4.4
	QL	6.2	5.6	4.4	4.0	6	4.5	5.6	5.4
	EL	4.5	5.8	4.6	5.5	6.3	5	5.7	5
	SR	5.1	5.8	5.2	5.0	6.2	5.5	5	4.8
120	ZLST	5.4	4.8	4.8	5.8	5.1	4.1	4.7	4.9
	QL	5.5	5.4	4.5	5.1	5	5.1	5.4	5.2
	EL	4.8	4.5	4.0	5.2	6.2	4.3	5.1	6.6
	SR	5.1	5.3	4.3	5.5	5.6	4.2	3.8	6

S.5.2. *Additional numerical comparison.* As suggested by one reviewer, we provide some numerical comparisons of the proposed tests with the one proposed by [13]. In this comparison, we adapt the settings of Example 4.4 in [13] for the compound symmetry structure of covariance matrix. As code from the first author of [13], repeated measurements were generated from the following linear model:

$$Y_{ij} = \mathbf{x}_{ij}^T\boldsymbol{\beta} + \varepsilon_{ij},\ i = 1, \dots, n,\ j = 1, \cdots, p$$

where $\boldsymbol{\beta} = (0, 0)^T$ and $\mathbf{x}_{ij} = (X_{ij1}, X_{ij2})^T$ is a 2×1 covariate vector, and $\mathbf{X}_{ik} = (X_{i1k}, \cdots, X_{ipk})^T$, $k = 1$ and 2 are independently generated from the multivariate normal distribution with mean 0 and covariance matrix $(\sigma 0.6^{|k-l|})_{p\times p}$, $k, l = 1, \cdots, p$. Our goal is to test the covariance structure of random error vector $\boldsymbol{\varepsilon}_i$. The random errors $\boldsymbol{\varepsilon}_i = (\varepsilon_{i1}, \cdots, \varepsilon_{ip})^T$ are generated by

$$\varepsilon_{ij} = \nu\varepsilon_{i,j-1} + \eta_i + u_{ij},\ j = 1, \dots, p \text{ and } i = 1, \dots, n, \tag{S.1}$$

where ε_{i0} follows $\mathcal{N}(0, 1 - \rho^2)$, u_{ij} follows $\mathcal{N}(0, (1 - \nu^2))$ and η_i follows $\mathcal{N}(0, \rho^2)$ with $\rho^2 < 1$ and $\nu^2 < 1$. u_{ij}, η_i and ε_{i0} are mutually independent. When $\nu = 0$, the covariance matrix Σ of $\boldsymbol{\varepsilon}_i = (\varepsilon_{i1}, \cdots, \varepsilon_{ip})^T$ is $\mathbf{I}_p + \rho^2\mathbf{1}_p\mathbf{1}_p^T$, which is a compound symmetric matrix. Thus, we may use the proposed tests and the one proposed by [13] for

$$H_0 : \Sigma = \mathbf{I}_p + \rho^2\mathbf{1}_p\mathbf{1}_p^T$$

Table S.6
Empirical power (in percentage) for $\nu = 0.1$ and 0.3 at level $\alpha = 0.05$

n	Method	$p = 270$	385	500	625	670	785	900	1025
					$\nu = 0.1$				
60	ZLST	2.1	1.7	0.3	0.7	0.6	0.2	0.9	0.6
	QL	13.9	9.8	11.7	10.1	10.2	10.4	11.6	10.1
	EL	11.6	8.5	8.4	7.3	7.7	6.7	5.9	4.5
	SR	6.7	5.5	6.2	5.9	6.5	5.2	6.4	5.6
80	ZLST	2.5	2.9	1.1	1.2	1.3	0.7	0.9	0.8
	QL	14	14.4	15.2	15.9	13.6	14.1	14.9	12.5
	EL	10.7	12	12.9	13.3	9.5	11.6	10	7
	SR	7.9	8.6	8.4	8.7	7.8	7.8	6.9	7.6
100	ZLST	3.9	3.9	2.8	1.6	2.2	1.4	1.4	0.9
	QL	17.9	17.9	19.3	17.1	18.4	17.4	17.3	17.8
	EL	14.3	14.3	17.2	14.4	15.1	13.6	13.8	13.9
	SR	9.2	9.2	10.9	10.3	11.5	9.6	8	10.8
120	ZLST	5.4	3.5	2.2	2.4	2.3	1.8	1.1	1.2
	QL	22	25.3	21.9	23.4	23.1	24	23	22.9
	EL	16.3	18.4	19.3	18.6	19.8	21.1	17.6	18.3
	SR	11.8	12.8	10.8	12.1	12.4	13	13.1	12.2
					$\nu = 0.3$				
60	ZLST	6.2	5.5	3.8	3.1	3	2.2	1.7	2.2
	QL	100	100	99.8	100	99.9	100	100	99.9
	EL	99.9	99.9	99.9	99.9	99.8	100	99.9	99.7
	SR	93	94.9	95.2	93.4	93.8	94.4	94.5	94.4
80	ZLST	14	10.1	7.2	5.3	5.5	4.4	3.7	4.2
	QL	100	100	100	100	100	100	100	100
	EL	100	100	100	100	100	100	100	100
	SR	99.8	99.5	99.8	99.7	99.8	99.9	99.8	99.8
100	ZLST	24	15.9	12	8.3	7.9	6.3	6.2	6.1
	QL	100	100	100	100	100	100	100	100
	EL	100	100	100	100	100	100	100	100
	SR	100	100	100	100	100	100	100	100
120	ZLST	36	24.9	18.6	14.1	14.1	10.6	8.9	6.3
	QL	100	100	100	100	100	100	100	100
	EL	100	100	100	100	100	100	100	100
	SR	100	100	100	100	100	100	100	100

Table S.7
Empirical power (in percentage) for $\nu = 0.5$ and 0.7 at level $\alpha = 0.05$

n	Method	$p = 270$	385	500	625	670	785	900	1025
					$\nu = 0.5$				
60	ZLST	52.7	38.7	31.4	22.5	24.2	16.8	16.5	14.7
	QL	100	100	100	100	100	100	100	100
	EL	100	100	100	100	100	100	100	100
	SR	100	100	100	100	100	100	100	100
80	ZLST	77.4	63.7	51.3	41.8	42.2	37.8	28.6	26.5
	QL	100	100	100	100	100	100	100	100
	EL	100	100	100	100	100	100	100	100
	SR	100	100	100	100	100	100	100	100
100	ZLST	92.3	83.4	71.7	62.3	58	51.1	47.8	40.4
	QL	100	100	100	100	100	100	100	100
	EL	100	100	100	100	100	100	100	100
	SR	100	100	100	100	100	100	100	100
120	ZLST	98.7	93.3	86.8	78.1	75.6	69.8	61.3	57.2
	QL	100	100	100	100	100	100	100	100
	EL	100	100	100	100	100	100	100	100
	SR	100	100	100	100	100	100	100	100
					$\nu = 0.7$				
60	ZLST	98.5	95.7	93	86.7	86.2	79	74.5	70.7
	QL	100	100	100	100	100	100	100	100
	EL	100	100	100	100	100	100	100	100
	SR	100	100	100	100	100	100	100	100
80	ZLST	100	99.8	99.1	97.8	97.2	96.5	93.8	88.7
	QL	100	100	100	100	100	100	100	100
	EL	100	100	100	100	100	100	100	100
	SR	100	100	100	100	100	100	100	100
100	ZLST	100	100	100	99.8	99.7	99.1	98.7	97.9
	QL	100	100	100	100	100	100	100	100
	EL	100	100	100	100	100	100	100	100
	SR	100	100	100	100	100	100	100	100
120	ZLST	100	100	100	100	100	99.9	99.9	99.8
	QL	100	100	100	100	100	100	100	100
	EL	100	100	100	100	100	100	100	100
	SR	100	100	100	100	100	100	100	100

In this example, we take $\rho = 0.5$, and we set $\nu = 0$ to study whether the tests can retain Type I error rate, and $\nu = 0.1, 0.3, 0.5$ and 0.7 to study the empirical powers. In this numerical comparison, we take $n = 60, 80, 100$ and 120, and $p = 270$, 385, 500, 625, 670, 785, 900 and 1025. For each case, we run 1000 replications. Thus, the Monte Carlo error rate is approximately 1.35% for the significance level 0.05

We compare the proposed QL-test and EL-test, the test proposed by Srivastava and Reid (2012), denoted by SR, and the test proposed in [13], denoted by ZLST, in the tables. The empirical size (Type I error rate) at level $\alpha = 0.05$ of these four tests are presented in Table S.5, from which we can see that all these tests retain correct sizes.

The empirical powers of these four tests at level 0.05 are presented in Tables S.6 and S.7, from which we can see that QL-test performs the best, EL-test performs better than SR, and SR performs better than ZLST. All empirical powers of these four tests increase as the sample size or ν increases, and decrease as the dimension increases. Tables S.6 also shows that the empirical power of ZLST can be significantly lower than the significance level 0.05. This is not desirable.

S.5.3. *Real data example.*

Example S.1 We illustrate the proposed tests by an empirical analysis of a real data set. The data are extracted from a commercial database containing weekly returns for some stocks traded in the Chinese stock market during the period from September 1, 2006 to December 25, 2009. After data cleaning and excluding observations with missing values, we obtain a data subset containing 132 weekly returns for 97 stocks. It is of great interest to examine whether the Chinese stock market returns follow two classical pricing models: the capital asset pricing model (CAPM [7, 10, 6]) and the Fama and French's three-factor model (TFM [3, 4]). To this end, we formulate this problem as testing that the covariance matrix of the residual vector of the fitted pricing model satisfies a structure. Specifically, we consider three factors: z_1 stands for weekly returns of the Shanghai Composite Index (i.e., the market index); z_2 for the difference in returns between portfolios of small capitalization firms and large capitalization firms and z_3 for the difference in returns between portfolios of high book-to-market ratio firms and low book-to-market ratio firms. The CAPM includes only the factor z_1 under the efficient market assumption. Let $\mathbf{y}_i$ be a 97-dimensional row vector for weekly returns of 97 stocks at the ith week. The CAPM follows the multi-response regression model

$$\mathbf{y}_i = \mathbf{b}_{10} + \mathbf{b}_{11} z_{1i} + \mathbf{e}_{1i}, \tag{S.2}$$

where $\mathbf{b}_{10}, \mathbf{b}_{11}$ are the regression coefficient vectors and $\mathbf{e}_{1i}$ is the random error. Let $\mathbf{z}_i = (z_{1i}, z_{2i}, z_{3i})^T$. The TFM follows

$$\mathbf{y}_i = \mathbf{b}_{20} + \mathbf{b}_{21}\mathbf{z}_i + \mathbf{e}_{2i}. \tag{S.3}$$

We first examine whether the CAPM is sufficient for deciding the stock returns. This can be formulated to test the correlation matrix of the corresponding residual vector having the structure: $\mathbf{I}_p + \theta(\mathbf{1}_p\mathbf{1}_p^T - \mathbf{I})$. We apply the proposed methods to test

$$\mathbf{H}_{20a}: \ \mathrm{Corr}(\mathbf{e}_{1i}) = \mathbf{I}_p + \theta_1(\mathbf{1}_p\mathbf{1}_p^T - \mathbf{I}_p)$$

The estimator of coefficient is $\hat{\theta}_1 = 0.0988$ and $\hat{\kappa} = 3.0003$. The EL-test and QL-test statistics are 15.7145 and 35.9279, respectively. Both EL-test and QL-test reject the null hypothesis with the p-values 0.00 and 0.00, respectively. This implies that the residuals are not equally correlated and $\theta_1 \neq 0$. The CAPM model was proposed based on the assumption of efficient market and all stock returns should be on the efficient frontier. Both EL-test and QL-test implies that Chinese market is not efficient.

We next examine whether TFM can describe the stock pricing well by testing

$$\mathbf{H}_{20b}: \ \mathrm{Corr}(\mathbf{e}_{2i}) = \mathbf{I}_p + \theta_2(\mathbf{1}_p\mathbf{1}_p^T - \mathbf{I}_p)$$

The estimator is $\hat{\theta}_2 = 0.0306$ which is less than $\hat{\theta}_1$ and $\hat{\kappa} = 3.0001$. This implies that the TFM indeed improves the results. The EL-test and QL-test statistics are 14.5898 and 26.9419, respectively. Both EL-test and QL-test also reject the null hypothesis H_{20b} with the p-values 0.00 and 0.00, respectively.

In recent years, [2] suggests to introducing the fourth factor, the monthly premium on winners minus losers, into the Fama-French's three-factor model. This model is usually named the Carhart's four-factor model (CFM) and defined by

$$\mathbf{y}_i = \mathbf{b}_{30} + \mathbf{b}_{31}\mathbf{z}_i + \mathbf{e}_{3i}, \tag{S.4}$$

where $\mathbf{z}_i = (z_{1i}, z_{2i}, z_{3i}, z_{4i})^T$. Analogously, we consider the hypothesis testing

$$\mathbf{H}_{30b}: \ \mathrm{Corr}(\mathbf{e}_{3i}) = \mathbf{I}_p + \theta_3(\mathbf{1}_p\mathbf{1}_p^T - \mathbf{I}_p)$$

The estimator is $\hat{\theta}_3 = 0.0297$ which almost equals $\hat{\theta}_2$ and $\hat{\kappa}$ also is 3.0001. This implies that the CFM barely improves the results from the TFM. The EL-test and QL-test statistics are 15.7732 and 27.8011, respectively. We have the same result that both two tests reject the null hypothesis.

This empirical analysis seems to imply that the two classical pricing models do not describe the Chinese stock returns sufficiently.

S.6. Numerical computations of $(F_1^{y_{n-1},G}, \mu_1^{(1)}, \sigma_{n1}^{(1)})$ and $(F_2^{y_{n-1},G}, \mu_2^{(1)}, \sigma_{n2}^{(1)})$. The asymptotical means and variances involve integrals along paths in the complex plane. As known, some integrals along paths in the complex plane may be calculated by using Cauchy's residue theorem (Chapter 10 in [9]), while others may be calculated by using numerical methods. First of all, we do not need the expressive forms of the asymptotical means and variances under H_1 in the implementation of the proposed test since only the limiting null distributions are needed. However, they are useful for calculating the power functions of the proposed QL- and EL-tests. Thus, we present a numerical algorithm to calculate these asymptotic mean and variances.

Let $\lambda_{10}, ..., \lambda_{p0}$ be the eigenvalues of $\mathbf{\Lambda} = \mathbf{\Gamma}^T \mathbf{B}_* \mathbf{\Gamma}$. Let $G(t)$ be the limiting spectral distribution of $\mathbf{\Gamma}^T \mathbf{B}_* \mathbf{\Gamma}$. Denote $\lambda_{\min}(\mathbf{A})$ and $\lambda_{\max}(\mathbf{A})$ to be the minimum and maximum eigenvalues of $\mathbf{A}$, respectively. Define

$$a = (1 - \sqrt{p/(n-1)})^2 \lambda_{\min}(\mathbf{\Gamma}^T \mathbf{B}_* \mathbf{\Gamma}),\ b = (1 + \sqrt{p/(n-1)})^2 \lambda_{\max}(\mathbf{\Gamma}^T \mathbf{B}_* \mathbf{\Gamma})$$

Let $\underline{m}_{\Re}(z)$ and $\underline{m}_{\Im}(z)$ be the real part and imaginary part of $\underline{m}(z)$ with $z = x + \mathbf{i}y$, respectively, where $\mathbf{i}^2 = -1$. The relation between z and $\underline{m}(z)$ is given in Lemma S.2:

$$z = -\frac{1}{\underline{m}(z)} + c \int \frac{t}{1 + t m(z)}\, dG(t). \tag{S.1}$$

where we use $c = \lim_{n\to\infty} p/(n-1)$ instead of using y that is used in the text to avoid the confusion with the imaginary part of z. (S.1) implies that

$$x = \frac{-\underline{m}_{\Re}(z)}{\underline{m}_{\Re}^2(z) + \underline{m}_{\Im}^2(z)} + c \int \frac{t + t^2 \underline{m}_{\Re}(z)}{(1 + t\underline{m}_{\Re}(z))^2 + t^2 \underline{m}_{\Im}^2(z)}\, dG(t), \tag{S.2}$$

$$y = \frac{\underline{m}_{\Im}(z)}{\underline{m}_{\Re}^2(z) + \underline{m}_{\Im}^2(z)} + c \int \frac{-t^2 \underline{m}_{\Im}(z)}{(1 + t\underline{m}_{\Re}(z))^2 + t^2 \underline{m}_{\Im}^2(z)}\, dG(t). \tag{S.3}$$

In the numerical approximation, we will generate the lattice points of (x, y) and then obtain $(\underline{m}_{\Re}(z), \underline{m}_{\Im}(z))$ by using (S.2) and (S.3). Thus, we can obtain $\underline{m}(z)$ that is used for numerical integration. In our numerical algorithm, we approximate c by p/n and

$$\int h(t; z)\, dG(t) = \frac{1}{p} \sum_{j=1}^{p} h(\lambda_{j0}; z)$$

in viewing a general function $h(t; z)$ as a function of t. Thus, in our numerical algorithm $\underline{m}_{\Re}(z)$ and $\underline{m}_{\Im}(z)$ can be obtained by the following system of two equations

$$x \approx \frac{-\underline{m}_{\Re}(z)}{\underline{m}_{\Re}^2(z)+\underline{m}_{\Im}^2(z)} + \frac{y_{n-1}}{p}\sum_{j=1}^{p}\frac{\lambda_{j0}+\lambda_{j0}^2\underline{m}_{\Re}(z)}{(1+\lambda_{j0}\underline{m}_{\Re}(z))^2+\lambda_{j0}^2\underline{m}_{\Im}^2(z)},$$

$$y \approx \frac{\underline{m}_{\Im}(z)}{\underline{m}_{\Re}^2(z)+\underline{m}_{\Im}^2(z)} + \frac{y_{n-1}}{p}\sum_{j=1}^{p}\frac{-\lambda_{j0}^2\underline{m}_{\Im}(z)}{(1+\lambda_{j0}\underline{m}_{\Re}(z))^2+\lambda_{j0}^2\underline{m}_{\Im}^2(z)}. \tag{S.4}$$

This coincides to (4) of [5]. Thus, for a given (x, y), we can obtain $(\underline{m}_{\Re}(z), \underline{m}_{\Im}(z))$ by solving (S.4) by the Newton method.

The point at which $\underline{m}(z)$ has singularity lies over the interval $[a, b]$. Thus, we need to generate a path $\mathcal{C}_1$ whose interior covers $[a, b]$ for calculating the contour integration involved in the asymptotical means and variances. Since the asymptotical variances involve two-fold contour integration, we need to create another path $\mathcal{C}_2$ that does not intersect with $\mathcal{C}_1$. From theory of complex analysis, the shape or the size of the paths $\mathcal{C}_1$ and $\mathcal{C}_2$ do not matter, provided that (a) they cover the singularity point and (b) they do not intersect. To this end, we set $\varepsilon_2 = (b-a)/N$ and $\varepsilon_1 = \varepsilon_2/10$ for a given large integer N to be specified in our numerical algorithm, and

$$\begin{aligned}\mathcal{C}_1 \quad = \quad & \{z = x \pm \epsilon_1 \mathbf{i}, x \in [a, b]\} \bigcup \{z = b + y\mathbf{i}, y \in [-\epsilon_1, \epsilon_1]\} \\ & \bigcup \{z = a + y\mathbf{i}, y \in [-\epsilon_1, \epsilon_1]\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{C}_2 \quad = \quad & \{z = x \pm \epsilon_2 \mathbf{i}, x \in [a-\epsilon_2, b+\epsilon_2]\} \bigcup \{z = b + \epsilon_2 + y\mathbf{i}, y \in [-\epsilon_2, \epsilon_2]\} \\ & \bigcup \{z = a - \epsilon_2 + y\mathbf{i}, y \in [-\epsilon_2, \epsilon_2]\}.\end{aligned}$$

Thus, $\{z = x + 0\mathbf{i} : x \in [a, b]\}$ lies interior of $\mathcal{C}_1$ and $\mathcal{C}_2$ and $\mathcal{C}_1 \bigcap \mathcal{C}_2$ is an empty set. Set $\Delta = (b-a)/N$ (i.e., $\Delta = \varepsilon_2$), and

$$\begin{aligned} x_k &= a + k\Delta, \quad k = 0, ..., N \\ \tilde{x}_k &= a - \Delta + k\Delta, \quad k = 0, ..., N+2, \end{aligned}$$

which equally partition $[a, b]$ and $[a - \varepsilon_2, b + \varepsilon_2]$, respectively.

Let $z_{1s} = x_s + \mathbf{i}\epsilon_1$ for $s = 0, ..., N$ and $z_{2t} = \tilde{x}_t + \mathbf{i}\epsilon_2$ for $t = 0, ..., N+2$.

By Lemma 1 and Algorithm 2 of [5], we have

$$\int f_\ell(x) dF^{y_{n-1},G}(x) \approx \frac{\Delta}{y_{n-1}\pi}\sum_{s=0}^{N} f_\ell(x_s)\underline{m}_{\Im}(z_{1s}), \tag{S.5}$$

and let $f_\ell'(z)$ be the derivative of $f_\ell(z)$ about z, it follows that

$$
\begin{aligned}
&-\frac{1}{2\pi\mathbf{i}}\oint_{\mathcal{C}_1} f_\ell(z)\frac{y_{n-1}\int \underline{m}^3(z)t^2(1+t\underline{m}(z))^{-3}dG(t)}{(1-y_{n-1}\int \underline{m}^2(z)t^2(1+t\underline{m}(z))^{-2}dG(t))^2}dz\\
=&\frac{1}{2\pi}\int_a^b f_\ell'(x)\arg\left(1-y_{n-1}\int\frac{t^2\underline{m}^2(x)}{(1+t\underline{m}(x))^2}dG(t)\right)dx \qquad \text{(S.6)}\\
\approx&\frac{\Delta}{2\pi}\sum_{s=0}^{N} f_\ell'(x_s)\arg\left(1-\frac{y_{n-1}}{p}\sum_{j=1}^{p}\frac{\lambda_{j0}^2\underline{m}^2(z_{1s})}{(1+\lambda_{j0}\underline{m}(z_{1s}))^2}\right),
\end{aligned}
$$

where $\arg(z)$ stands for the angle of z (i.e., $\arctan(y/x)$ for $z=x+y\mathbf{i}$), and

$$
\begin{aligned}
&-\frac{(\kappa-3)}{2\pi\mathbf{i}}\oint_{\mathcal{C}_1} f_\ell(z)\frac{y_{n-1}\int \underline{m}^3(z)t^2(1+t\underline{m}(z))^{-3}dG(t)}{1-y_{n-1}\int \underline{m}^2(z)t^2(1+t\underline{m}(z))^{-2}dG(t)}dt\\
=&\frac{(\kappa-3)}{2\pi}\int_a^b f_\ell'(x)\Im\left(1-y_{n-1}\int\frac{t^2\underline{m}^2(x)}{(1+t\underline{m}(x))^2}dG(t)\right)dx \qquad \text{(S.7)}\\
\approx&-\frac{(\kappa-3)\Delta}{2\pi}\sum_{s=0}^{N} f_\ell'(x_s)\Im\left(1-\frac{y_{n-1}}{p}\sum_{j=1}^{p}\frac{\lambda_{j0}^2\underline{m}^2(z_{1s})}{(1+\lambda_{j0}\underline{m}(z_{1s}))^2}\right).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&-\frac{1}{2\pi^2}\oint\oint\frac{f_\ell(z_1)f_\ell(z_2)}{(\underline{m}(z_1)-\underline{m}(z_2))^2}d\underline{m}(z_1)d\underline{m}(z_2)\\
=&\frac{1}{2\pi^2}\int_{a-\epsilon_2}^{b+\epsilon_2}\int_a^b f_\ell'(x)f_\ell'(y)\log\left(1+4\frac{\underline{m}_\Im(x)\underline{m}_\Im(y)}{|\underline{m}(x)-\underline{m}(y)|^2}\right)dxdy \qquad \text{(S.8)}\\
\approx&\frac{\Delta^2}{2\pi^2}\sum_{s=0}^{N}\sum_{t=0}^{N+2} f_\ell'(x_s)f_\ell'(\tilde{x}_t)\log\left(1+4\frac{\underline{m}_\Im(z_{1s})\underline{m}_\Im(z_{2t})}{|\underline{m}(z_{1s})-\underline{m}(z_{2t})|^2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{y_{n-1}(\kappa-3)}{4\pi^2}\int\int\left(\int\frac{f_\ell(z_1)f_\ell(z_2)t^2}{(\underline{m}(z_1)t+1)^2(\underline{m}(z_2)t+1)^2}dG(t)\right)d\underline{m}(z_1)d\underline{m}(z_2)\\
\approx&-\frac{y_{n-1}(\kappa-3)\Delta^2}{p\pi^2}\sum_{j=1}^{p}\Re\left(\sum_{s=0}^{N}\sum_{t=0}^{N+2}\frac{f_\ell'(z_{1s})f_\ell'(z_{2t})}{(1+\lambda_{j0}\underline{m}(z_{1s}))(1+\lambda_{j0}\underline{m}(z_{2t}))}\right).
\end{aligned}
$$

(S.9)

Now we summarize the above discussions as a numerical algorithm below.

An Algorithm

Step 1. To obtain $(\underline{m}_{\Re}(z_{1s}), \underline{m}_{\Im}(z_{1s}))$ and $(\underline{m}_{\Re}(z_{2t}), \underline{m}_{\Im}(z_{2t}))$: By (S.4), we have

$$x_s \approx \frac{-\underline{m}_{\Re}(z_{1s})}{\underline{m}^2_{\Re}(z_{1s}) + \underline{m}^2_{\Im}(z_{1s})} + \frac{y_{n-1}}{p} \sum_{j=1}^{p} \frac{\lambda_{j0} + \lambda^2_{j0}\underline{m}_{\Re}(z_{1s})}{(1+\lambda_{j0}\underline{m}_{\Re}(z_{1s}))^2 + \lambda^2_{j0}\underline{m}^2_{\Im}(z_{1s})},$$
$$\epsilon_1 \approx \frac{\underline{m}_{\Im}(z_{1s})}{\underline{m}^2_{\Re}(z_{1s}) + \underline{m}^2_{\Im}(z_{1s})} + \frac{y_{n-1}}{p} \sum_{j=1}^{p} \frac{-\lambda^2_{j0}\underline{m}_{\Im}(z_{1s})}{(1+\lambda_{j0}\underline{m}_{\Re}(z_{1s}))^2 + \lambda^2_{j0}\underline{m}^2_{\Im}(z_{1s})}. \tag{S.10}$$

for $s = 0, ..., N$. Then by the Newton method, we obtain the solution $(\underline{m}_{\Re}(z_{1s}), \underline{m}_{\Im}(z_{1s}))$ of the two equations (S.10). Similarly, we have

$$\tilde{x}_t \approx \frac{-\underline{m}_{\Re}(z_{2t})}{\underline{m}^2_{\Re}(z_{2t}) + \underline{m}^2_{\Im}(z_{2t})} + \frac{y_{n-1}}{p} \sum_{j=1}^{p} \frac{\lambda_{j0} + \lambda^2_{j0}\underline{m}_{\Re}(z_{2t})}{(1+\lambda_{j0}\underline{m}_{\Re}(z_{2t}))^2 + \lambda^2_{j0}\underline{m}^2_{\Im}(z_{2t})},$$
$$\epsilon_2 \approx \frac{\underline{m}_{\Im}(z_{2t})}{\underline{m}^2_{\Re}(z_{2t}) + \underline{m}^2_{\Im}(z_{2t})} + \frac{y_{n-1}}{p} \sum_{j=1}^{p} \frac{-\lambda^2_{j0}\underline{m}_{\Im}(z_{2t})}{(1+\lambda_{j0}\underline{m}_{\Re}(z_{2t}))^2 + \lambda^2_{j0}\underline{m}^2_{\Im}(z_{2t})}. \tag{S.11}$$

for $t = 0, ..., N+2$. Then by the Newton method, we obtain the solution $(\underline{m}_{\Re}(z_{2t}), \underline{m}_{\Im}(z_{2t}))$ of the two equations (S.11).

Step 2. To $(F_1^{y_{n-1},G}, \mu_1^{(1)}, \sigma_{n1}^{(1)})$ of the function $f_1(z) = z - \log z - 1$. Thus, by (S.5)—(S.9), as $p < n-1$, we have

$$F_1^{y_{n-1},G} = \int f_1(x) dF^{y_{n-1},G}(x) \approx \frac{\Delta}{y_{n-1}\pi} \sum_{s=0}^{N} (x_s - \log x_s - 1)\underline{m}_{\Im}(z_{1s}),$$

$$
\begin{aligned}
& \mu_1^{(1)} \\
= \quad & \log|\mathbf{B}_*\boldsymbol{\Sigma}_1^*| + \mathrm{tr}\boldsymbol{\Sigma}\boldsymbol{\Sigma}_1^{*-1} - p \\
& -\frac{1}{2\pi\mathbf{i}}\oint_{\mathcal{C}_1} f_1(z)\frac{y_{n-1}\int \underline{m}^3(z)t^2(1+t\underline{m}(z))^{-3}dG(t)}{(1-y_{n-1}\int \underline{m}^2(z)t^2(1+t\underline{m}(z))^{-2}dG(t))^2}dz \\
& -\frac{(\kappa-3)}{2\pi\mathbf{i}}\oint_{\mathcal{C}_1} f_1(z)\frac{y_{n-1}\int \underline{m}^3(z)t^2(1+t\underline{m}(z))^{-3}dG(t)}{1-y_{n-1}\int \underline{m}^2(z)t^2(1+t\underline{m}(z))^{-2}dG(t)}dt \\
\approx \quad & \log|\mathbf{B}_*\boldsymbol{\Sigma}_1^*| + \mathrm{tr}\boldsymbol{\Sigma}\boldsymbol{\Sigma}_1^{*-1} - p \\
& +\frac{\Delta}{2\pi}\sum_{s=0}^{N}(1-1/x_s)\arg\left(1-\frac{y_{n-1}}{p}\sum_{j=1}^{p}\frac{\lambda_{j0}^2\underline{m}^2(z_{1s})}{(1+\lambda_{j0}\underline{m}(z_{1s}))^2}\right) \\
& -\frac{(\kappa-3)\Delta}{2\pi}\sum_{s=0}^{N}(1-1/x_s)\Im\left(1-\frac{y_{n-1}}{p}\sum_{j=1}^{p}\frac{\lambda_{j0}^2\underline{m}^2(z_{1s})}{(1+\lambda_{j0}\underline{m}(z_{1s}))^2}\right),
\end{aligned}
$$

$$
\begin{aligned}
& (\sigma_{n1}^{(1)})^2 \\
= \quad & -\frac{1}{2\pi^2}\oint\oint\frac{f_1(z_1)f_1(z_2)}{(\underline{m}(z_1)-\underline{m}(z_2))^2}d\underline{m}(z_1)d\underline{m}(z_2) \\
& -\frac{y_{n-1}(\kappa-3)}{4\pi^2}\int\int\left(\int\frac{f_1(z_1)f_1(z_2)t^2}{(\underline{m}(z_1)t+1)^2(\underline{m}(z_2)t+1)^2}dG(t)\right)d\underline{m}(z_1)d\underline{m}(z_2) \\
\approx \quad & \frac{\Delta^2}{2\pi^2}\sum_{s=0}^{N}\sum_{t=0}^{N+2}(1-1/x_s)(1-1/\tilde{x}_t)\log\left(1+4\frac{\underline{m}_{\Im}(z_{1s})\underline{m}_{\Im}(z_{2t})}{|\underline{m}(z_{1s})-\underline{m}(z_{2t})|^2}\right) \\
& -\frac{y_{n-1}(\kappa-3)\Delta^2}{\pi^2 p}\sum_{j=1}^{p}\Re\left(\sum_{s=0}^{N}\sum_{t=0}^{N+2}\frac{(1-1/z_{1s})(1-1/z_{2t})}{(1+\lambda_{j0}\underline{m}(z_{1s}))(1+\lambda_{j0}\underline{m}(z_{2t}))}\right).
\end{aligned}
$$

where $\lambda_{10},...,\lambda_{p0}$ are the eigenvalues of $\boldsymbol{\Lambda}=\boldsymbol{\Gamma}^T\mathbf{B}_*\boldsymbol{\Gamma}$.

Step 3. To obtain $(F_2^{y_{n-1},G}, \mu_2^{(1)}, \sigma_{n2}^{(1)})$ of the function $f_2(z) = y_{n-1}^{-1}z - \log z - 1$.

Thus, by (S.5)—(S.9), as $p > n-1$, we have

$$
F_2^{y_{n-1},G} = \int f_2(x)dF^{y_{n-1},G}(x) \approx \frac{\Delta}{y_{n-1}\pi}\sum_{s=0}^{N}(y_{n-1}^{-1}x_s - \log x_s - 1)\underline{m}_{\Im}(z_{1s}),
$$

$$
\begin{aligned}
&\mu_2^{(1)}\\
= \;& \log|\mathbf{B}_*\mathbf{\Sigma}_1^*| + y_{n-1}^{-1}\mathrm{tr}\mathbf{\Sigma}\mathbf{\Sigma}_1^{*-1} - p\\
& -\frac{1}{2\pi\mathbf{i}}\oint_{\mathcal{C}_1} f_2(z)\frac{y_{n-1}\int \underline{m}^3(z)t^2(1+t\underline{m}(z))^{-3}dG(t)}{(1-y_{n-1}\int \underline{m}^2(z)t^2(1+t\underline{m}(z))^{-2}dG(t))^2}dz\\
& -\frac{(\kappa-3)}{2\pi\mathbf{i}}\oint_{\mathcal{C}_1} f_2(z)\frac{y_{n-1}\int \underline{m}^3(z)t^2(1+t\underline{m}(z))^{-3}dG(t)}{1-y_{n-1}\int \underline{m}^2(z)t^2(1+t\underline{m}(z))^{-2}dG(t)}dt\\
\approx \;& \log|\mathbf{B}_*\mathbf{\Sigma}_1^*| + y_{n-1}^{-1}\mathrm{tr}\mathbf{\Sigma}\mathbf{\Sigma}_1^{*-1} - p\\
& +\frac{\Delta}{2\pi}\sum_{s=0}^{N}\left(\frac{1}{y_{n-1}}-\frac{1}{x_s}\right)\arg\left(1-\frac{y_{n-1}}{p}\sum_{j=1}^{p}\frac{\lambda_{j0}^2\underline{m}^2(z_{1s})}{(1+\lambda_{j0}\underline{m}(z_{1s}))^2}\right)\\
& -\frac{(\kappa-3)\Delta}{2\pi}\sum_{s=0}^{N}\left(\frac{1}{y_{n-1}}-\frac{1}{x_s}\right)\Im\left(1-\frac{y_{n-1}}{p}\sum_{j=1}^{p}\frac{\lambda_{j0}^2\underline{m}^2(z_{1s})}{(1+\lambda_{j0}\underline{m}(z_{1s}))^2}\right),
\end{aligned}
$$

$$
\begin{aligned}
&(\sigma_{n2}^{(1)})^2\\
= \;& -\frac{1}{2\pi^2}\oint\oint\frac{f_2(z_1)f_2(z_2)}{(\underline{m}(z_1)-\underline{m}(z_2))^2}d\underline{m}(z_1)d\underline{m}(z_2)\\
& -\frac{y_{n-1}(\kappa-3)}{4\pi^2}\int\int\left(\int\frac{f_2(z_1)f_2(z_2)t^2}{(\underline{m}(z_1)t+1)^2(\underline{m}(z_2)t+1)^2}dG(t)\right)d\underline{m}(z_1)d\underline{m}(z_2)\\
\approx \;& \frac{\Delta^2}{2\pi^2}\sum_{s=0}^{N}\sum_{t=0}^{N+2}\left(\frac{1}{y_{n-1}}-\frac{1}{x_s}\right)\left(\frac{1}{y_{n-1}}-\frac{1}{\tilde{x}_t}\right)\log\left(1+4\frac{\underline{m}_{\Im}(z_{1s})\underline{m}_{\Im}(z_{2t})}{|\underline{m}(z_{1s})-\underline{m}(z_{2t})|^2}\right)\\
& -\frac{y_{n-1}(\kappa-3)\Delta^2}{\pi^2 p}\sum_{j=1}^{p}\Re\left(\sum_{s=0}^{N}\sum_{t=0}^{N+2}\frac{(1/y_{n-1}-1/z_{1s})(1/y_{n-1}-1/z_{2t})}{(1+\lambda_{j0}\underline{m}(z_{1s}))(1+\lambda_{j0}\underline{m}(z_{2t}))}\right)
\end{aligned}
$$

where $\lambda_{10},...,\lambda_{p0}$ are the eigenvalues of $\mathbf{\Lambda}=\mathbf{\Gamma}^T\mathbf{B}_*\mathbf{\Gamma}$.

As an illustration, we apply this algorithm to Example 3.1 with $W_j \sim N(0,1)$. Under H_0, we take $(\theta_1,\theta_2,\theta_3)=(6,1,0)$, we are able to calculate $(F_1^{y_{n-1},G},\mu_1^{(1)},\sigma_{n1}^{(1)})$ by hand, and it equals $(0.309, 0.349, 0.391)$ for $(n,p)=(100,200)$, and $(-0.198,-100.658,0.376)$ for $(n,p)=(200,100)$. While under H_1, we take $(\theta_1,\theta_2,\theta_3)=(6,1,0.5)$, we are unable to calculate $(F_2^{y_{n-1},G},\mu_2^{(1)},\sigma_{n2}^{(1)})$ by hand. To understand how the computational cost and numerical accuracy as N increases, we take $N = 500, 1000, 2000, 4000$ and 6000. The values obtained by the proposed numerical algorithm are depicted in Table S.8. It can be seen from Table S.8 that (a) the computational cost approximately linearly as N increases; (b) the numerical values quickly become stable as N

increases, and (3) the numerical values are very close to the true ones under H_0. In general, $N = 4000$ is large enough. Thus, we can obtain the numerical values of $(F_1^{y_{n-1},G}, \mu_1^{(1)}, \sigma_{n1}^{(1)})$ or $(F_2^{y_{n-1},G}, \mu_2^{(1)}, \sigma_{n2}^{(1)})$ within a few minutes. Table S.8 seems to imply that the computing time for $(n, p) = (100, 200)$ is less than $(n, p) = (200, 100)$ because the solutions of equations in Step 1 depend on eigenvalues λ_{j0}s.

TABLE S.8
Numerical values of $(F_1^{y_{n-1},G}, \mu_1^{(1)}, \sigma_{n1}^{(1)})$ *and* $(F_2^{y_{n-1},G}, \mu_2^{(1)}, \sigma_{n2}^{(1)})$ *for Example 3.1 and Computing time in seconds*

N	Under H_0	Time	Under H_1	Time
	$(F_1^{y_{n-1},G}, \mu_1^{(1)}, \sigma_{n1}^{(1)})$		$(F_2^{y_{n-1},G}, \mu_2^{(1)}, \sigma_{n2}^{(1)})$	
	$(n, p) = (100, 200)$		$(n, p) = (100, 200)$	
500	(0.312, 0.346, 0.364)	5	(0.324, 0.329, 0.371)	20
1000	(0.313, 0.349, 0.374)	10	(0.324, 0.343, 0.402)	38
2000	(0.313, 0.350, 0.380)	20	(0.324, 0.347, 0.419)	78
4000	(0.313, 0.350, 0.383)	40	(0.324, 0.349, 0.428)	163
6000	(0.313, 0.350, 0.384)	62	(0.324, 0.349, 0.432)	230
	$(n, p) = (200, 100)$		$(n, p) = (200, 100)$	
500	(-0.196, -100.644, 0.365)	18	(-0.192, -96.706, 0.314)	8
1000	(-0.196, -100.648, 0.373)	38	(-0.192, -96.726, 0.350)	16
2000	(-0.196, -100.649, 0.378)	73	(-0.193, -96.738, 0.371)	33
4000	(-0.196, -100.650, 0.380)	146	(-0.193, -96.742, 0.383)	66
6000	(-0.196, -100.651, 0.381)	221	(-0.193, -96.742, 0.388)	102

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